Multiple Correct Answer Type

1. The potential energy of a particle of mass m at a distance r from a fixed point O is given by $V(r) = kr^2/2$, where k is a positive constant of appropriate dimensions. This particle is moving in a circular orbit of radius R about the point O. If v is the speed of the particle and L is the magnitude of its angular momentum about O, which of the following statements is (are) true?

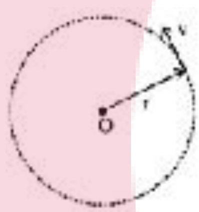
(A) $v = \sqrt{\frac{k}{2m}}R$

(B) $v = \sqrt{\frac{k}{m}}R$

(C) $L = \sqrt{mk}R^2$

(D) $L = \sqrt{\frac{mk}{2}}R^2$

Solution: (B, C)



$$V = \frac{kr^2}{2}$$

$F = -kr$ (towards center) $\left[F = -\frac{dV}{dr}\right]$

At $r = R$

$$v = \sqrt{\frac{kR^2}{m}} = \sqrt{\frac{k}{m}}R$$

$$L = \sqrt[m]{\frac{k}{m}}R^2$$

2. Consider a body of mass $1.0\ kg$ at rest at the origin at time $t = 0$ A force $\vec{F} = \left(\alpha t\hat{i} + \beta\hat{j}\right)$ is applied on the body, where $\alpha = 1.0Ns^{-1} and\ \beta = 1.0N$. The torque acting on the body about the origin at time $= 1.0\ s$ is $\vec{\tau}$. Which of the following statements is (are) true?

(A) $\left|\vec{\tau}\right| = \frac{1}{3}\ Nm$

(B) The torque $\vec{\tau}$ is in the direction of the unit vector $+\vec{k}$

(C) The velocity of the body at $t = 1\ s\ is\ \vec{v} = \frac{1}{2}\left(\hat{i} + 2\hat{j}\right)\ m\ s^{-1}$

(D) The magnitude of displacement of the body at $t = 1\ s$ is $\frac{1}{6}m$

Solution: (A, C)

$$\vec{F} = (\alpha t)\hat{i} + \beta\hat{j} \qquad \left[at\ t = 0, v = 0, \vec{r} = \vec{0}\right]$$

$$\alpha = 1, \beta = 1$$

$$\vec{F} = t\hat{i} + \hat{j}$$

$$m\frac{d\vec{v}}{dt} = t\hat{i} + \hat{j}$$

On integrating

$$m\vec{v} = \frac{t^2}{2}\hat{i} + t\hat{j} \qquad [m = 1kg]$$

$$\frac{d\vec{v}}{dt}\frac{t^2}{2}\hat{i} + t\hat{j} \qquad \left[\vec{r} = \vec{0} \text{ at } t = 0\right]$$

On integrating

$$\vec{r} = \frac{t^3}{6}\hat{i} + \frac{t^2}{2}\hat{j}$$

$$\vec{\tau} = -\frac{1}{3}\hat{k}$$

$$\vec{v} = \frac{t^2}{2}\hat{i} + t\hat{j}$$

At $t=1\ \vec{v}=\left(\frac{1}{2}\hat{i}+\hat{j}\right)=\frac{1}{2}\left(\hat{i}+2\hat{j}\right)m/sec$

At $t=1\ \vec{s}=\vec{r_1}-\vec{r_0}$

$$=\left[\frac{1}{6}\hat{i}+\frac{1}{2}\hat{j}\right]-\left[\vec{0}\right]$$

$$\vec{s}=\frac{1}{6}\hat{i}+\frac{1}{2}\hat{j}$$

$$\left|\vec{s}\right|=\sqrt{\left(\frac{1}{6}\right)^2+\left(\frac{1}{2}\right)^2}=\frac{\sqrt{10}}{6}m$$

3. A uniform capillary tube of inner radius r is dipped vertically into a beaker filled with water. The water rises to a height h in the capillary tube above the water surface in the beaker. The surface tension of water is σ. The angle of contact between water and the wall of the capillary tube is θ. Ignore the mass of water in the meniscus. Which of the following statements is (are) true?

(A) For a given material of the capillary tube, h decreases with increase in r

(B) For a given material of the capillary tube, h is independent of σ

(C) If this experiment is performed in a lift going up with a constant acceleration, then h decreases

(D) h is proportional to contact angle θ

Solution: (A, C)

$$\frac{2\sigma}{R}=\rho gh \qquad [R\rightarrow \text{Radius of meniscus}]$$

$h=\frac{2\sigma}{R\rho g}\qquad R=\frac{r}{\cos\theta}$ [r→radius of capillary; θ → contact angle]

$$h=\frac{2\sigma\cos\theta}{r\rho g}$$

(A) For given material, $\theta\rightarrow \text{constant}$

$$h\propto\frac{1}{r}$$

(B) h depend on σ

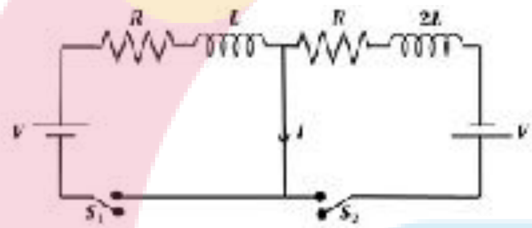
(C) If lift is going up with constant acceleration,

$g_{eff} = (g + a)$

$h = \dfrac{2\sigma\cos\theta}{r\rho(g+a)}$ It means h decreases

(D) h is proportional to $\cos\theta$ Not θ

4. In the figure below, the switches S_1 and S_2 are closed simultaneously at $t = 0$ and a current starts to flow in the circuit. Both the batteries have the same magnitude of the electromotive force (emf) and the polarities are as indicated in the figure. Ignore mutual inductance between the inductors. The current I in the middle wire reaches its maximum magnitude I_{max} at time $t = \tau$. Which of the following statements is (are) true?



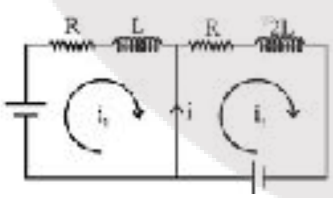
(A) $I_{max} = \dfrac{V}{2R}$

(B) $I_{max} = \dfrac{V}{4R}$

(C) $\tau = \dfrac{L}{R}\ln 2$

(D) $\tau = \dfrac{2L}{R}\ell\ln 2$

Solution: (B, D)



$i_{max} = (i_2 - i_1)_{max}$

$\Delta i = (i_2 - i_1) = \dfrac{V}{R}\left[1 - e^{-\left(\frac{R}{2L}\right)t}\right] - \dfrac{V}{R}\left[1 - e^{\left(-\frac{R}{L}\right)t}\right]$

$$\frac{V}{R}\left[e^{\left(-\frac{R}{L}\right)t} - e^{\left(-\frac{R}{2L}\right)t}\right]$$

For $(\Delta i)_{max} \frac{d(\Delta i)}{dt} = 0$

$$\frac{V}{R}\left[-\frac{R}{L}e^{-\left(\frac{R}{L}\right)t} - \left(-\frac{R}{2L}\right)e^{-\left(\frac{R}{2L}\right)t}\right] = 0$$

$$e^{-\left(\frac{R}{L}\right)t} = \frac{1}{2}e^{-\left(\frac{R}{2L}\right)t}$$

$$e^{-\left(\frac{R}{2L}\right)t} = \frac{1}{2}$$

$$\left(\frac{R}{2L}\right)t = \ell n2$$

$t = \frac{2L}{R}\ell n2 \rightarrow$ time when I is maximum

$$i_{max} = \frac{V}{R}\left[e^{-\frac{R}{L}\left(\frac{2L}{R}\ell n2\right)} - e^{-\left(\frac{R}{2L}\right)\left(\frac{2L}{R}\ell n2\right)}\right]$$

$$\left|i_{max}\right| = \frac{V}{R}\left|\left[\frac{1}{4} - \frac{1}{2}\right]\right| = \frac{1}{4}\frac{V}{R}$$

5. Two infinitely long straight wires lie in the xy-plane along the lines $x = \pm R$. The wire located at $x = +R$ carries a constant current I_1 and the wire located at $x = -R$ carries a constant current I_2. A circular loop of radius R is suspended with its centre at $\left(0, 0\sqrt{3}R\right)$ and in a plane parallel to the xy - plane. This loop carries a constant current I in the clockwise direction as seen from above the loop. The current in the wire is taken to be positive if it is in the $+\hat{j}$ direction. Which of the following statements regarding the magnetic field $\vec{B}$ is (are) true?

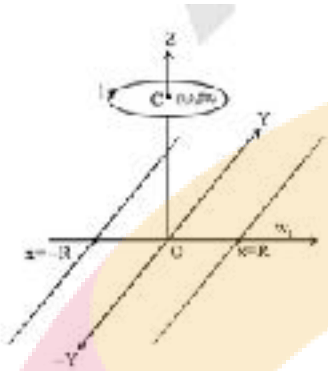
(A) If $I_1 = I_2$, then $\vec{B}$ cannot be equal to zero at the origin$(0, 0, 0)$

(B) If $I_1 > 0$ and $I_2 < 0$, then $\vec{B}$ can be equal to zero at the origin$(0, 0, 0)$

(C) If $I_1 < 0$ and $I_2 > 0$, then $\vec{B}$ can be equal to zero at the origin$(0, 0, 0)$

(D) If $I_1 = I_2$, Then the z-component of the magnetic field at the centre of the loop is $\left(-\frac{\mu_0 I}{2R}\right)$

Solution: (A, B, D)

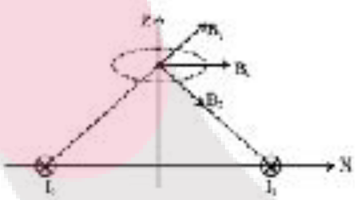


(A) At origin, $\vec{B} = 0$ due to two wires if $I_1 = I_2$, $hence$ $\left(\vec{B}_{net}\right)$at origin is equal to $\vec{B}$ due to ring, which is non-zero.

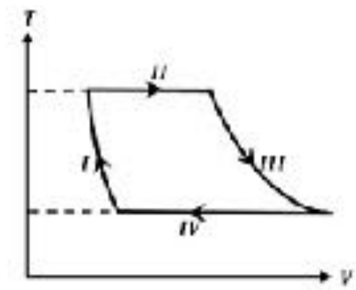
(B) If $I_1 > 0$ and $I_2 < 0, \vec{B}$ at origin due to wires will be along $+\hat{k}$ direction and $\vec{B}$ due to ring is along $-\hat{k}$ direction and hence $\vec{B}$ can be zero at origin

(C) If $I_1 < 0$ and $I_2 > 0, \vec{B}$ at origin due to wires is along $-\hat{k}$ and also along $-\hat{k}$ due to ring, hence $\vec{B}$ cannot be zero

(D)



6. One mole of a monatomic ideal gas undergoes a cyclic process as shown in the figure (where V is the volume and T is the temperature). Which of the statements below is (are) true?



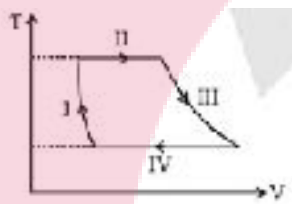
(A) Process I is an isochoric process

(B) In process II, gas absorbs heat

(C) In process IV, gas releases heat

(D) Processes I and III are not isobaric

Solution: (B, C, D)



(A) Process- I is not isochoric, V is decreasing

(B) Process- II is isothermal expansion

$\Delta U = 0, W > 0$

$\Delta Q > 0$

(C) Process-IV is isothermal compression

$\Delta U = 0, W < 0$

$\Delta Q < 0$

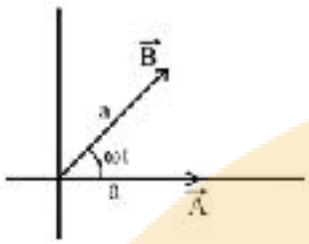
(D) Process-I and III are NOT isobaric because in isobaric process $T \propto V$ hence isobaric $T–V$ graph will be linear

Integer Answer Type

7. Two vectors $\vec{A}$ and $\vec{B}$ are defines as $\vec{A} = a\hat{i}\ and\ \vec{B} = a\left(\cos\omega t\ \hat{i} + \sin\omega t\hat{j}\right)$, where a is a

constant and $\omega = \frac{\pi}{6} rad\, s^{-1}$. If $\left|\vec{A} + \vec{B}\right| = \sqrt{3}\left|\vec{A} - \vec{B}\right|$ at time $t = \tau$ for the first time, the value of τ in seconds, is__

Solution: (2.00 sec)



$$\left|\vec{A} + \vec{B}\right| = 2\, a\cos\frac{\omega t}{2}$$

$$\left|\vec{A} - \vec{B}\right| = 2a\sin\frac{\omega t}{2}$$

So $2a\cos\frac{\omega t}{2} = \sqrt{3}\left(2a\sin\frac{\omega t}{2}\right)$

$$\tan\frac{\omega t}{2} = \frac{1}{\sqrt{3}}$$

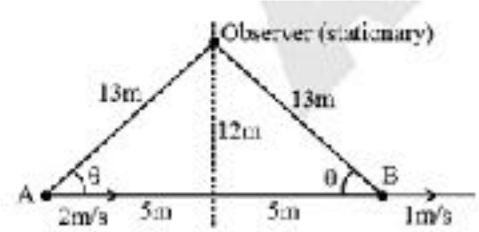
$$\frac{\omega t}{2} = \frac{\pi}{6} \Rightarrow \omega t = \frac{\pi}{3}$$

$$\frac{\pi}{6}t = \frac{\pi}{3}$$

$$t = 2.00\, sec$$

8. Two men are walking along a horizontal straight line in the same direction. The man in front walks at a speed $1.0\ m\, s^{-1}$ and the man behind walks at a speed $2.0\ m\, s^{-1}$. A third man is standing at a height $12\ m$ above the same horizontal line such that all three men are in a vertical plane. The two walking men are blowing identical whistles which emit a sound of frequency $1430\ Hz$. The speed of sound in air is $330\ m\, s^{-1}$. At the instant, when the moving men are $10\ m$ apart, the stationary man is equidistant from them. The frequency of beats in Hz, heard by the stationary man at this instant, is __________.

Solution: (5.00Hz)



$$\cos\theta = \frac{5}{13}$$

$$f_A = 1430\left[\frac{330}{330 - 2\cos\theta}\right] = 1430\left[\frac{1}{1 - \frac{2\cos\theta}{330}}\right] = 1430\left[1 + \frac{2\cos\theta}{330}\right]$$ (By binomial expansion)

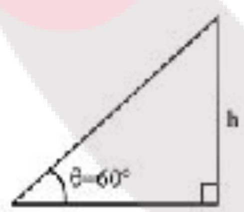
$$f_B = 1430\left[\frac{330}{330 + 1\cos\theta}\right] = 1430\left[1 - \frac{\cos\theta}{330}\right]$$

$$\Delta f = f_A - f_B = 1430\left[\frac{3\cos\theta}{330}\right] = 13\cos\theta$$

$$= 13\left(\frac{5}{13}\right) = 5.00Hz$$

9. A ring and a disc are initially at rest, side by side, at the top of an inclined plane which makes an angle 60^o with the horizontal. They start to roll without slipping at the same instant of time along the shortest path. If the time difference between their reaching the ground is $\left(2 - \sqrt{3}\right)/\sqrt{10}s$, then the height of the top of the inclined plane, in meters, is __________. Take $g = 10\ m\ s^{-2}$.

Solution: (0.75 m)



$$a_c = \frac{g\sin\theta}{1 + \frac{I_c}{MR^2}}$$

$$a_{ring} = \frac{g\sin\theta}{2}$$

$$a_{ring} = \frac{2g\sin\theta}{3}$$

$$\frac{h}{\sin\theta} = \frac{1}{2}\left(\frac{g\sin\theta}{2}\right)t_1^2 \Rightarrow t_1 = \sqrt{\frac{4h}{g\sin^2\theta}} = \sqrt{\frac{16h}{3g}}$$

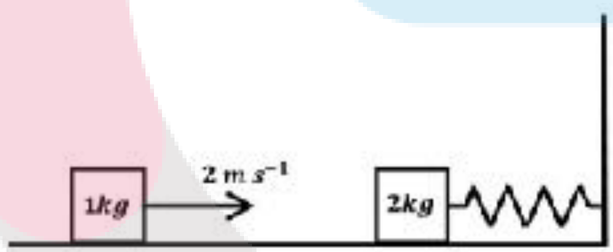
$$\frac{h}{\sin\theta} = \frac{1}{2}\left(\frac{2g\sin\theta}{3}\right)t_1^2 \Rightarrow t_2 = \sqrt{\frac{3h}{g\sin^2\theta}} = \sqrt{\frac{4h}{g}}$$

$$\Rightarrow \sqrt{\frac{16h}{3g}} - \sqrt{\frac{4h}{g}} = \frac{2-\sqrt{3}}{\sqrt{10}}$$

$$\sqrt{h}\left[\frac{4}{\sqrt{3}} - 2\right] = 2 - \sqrt{3}$$

$$\sqrt{h} = \frac{\left(2-\sqrt{3}\right)\sqrt{3}}{\left(4-2\sqrt{3}\right)} = \frac{\sqrt{3}}{2} \Rightarrow h = \frac{3}{4} = 0.75\ m$$

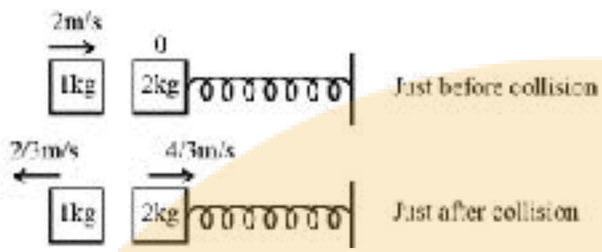
10. A spring - block system is resting on a frictionless floor as shown in the figure. The spring constant is $2.0\ N\,m^{-1}$ and the mass of the block is $2.0\ kg$. Ignore the mass of the spring. Initially the spring is in an unstretched condition. Another block of mass 1.0 kg moving with a speed of $2.0\ m\,s^{-1}$ collides elastically with the first block. The collision is such that the 2.0 kg block does not hit the wall. The distance, in $metres$, between the two blocks when the spring returns to its unstretched position for the first time after the collision is _________.



Solution: (2.09m)

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\ sec$$

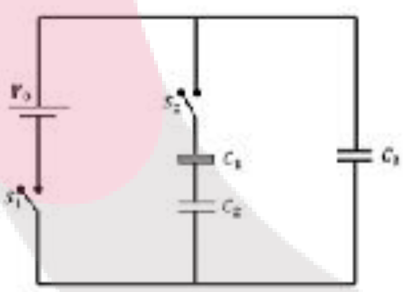
Block returns to original position in $\frac{T}{2} = \pi\ sec$



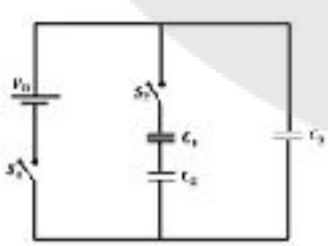
$$d = \frac{2}{3}(\pi) = \frac{2}{3}(3.14) = 2.0933m$$

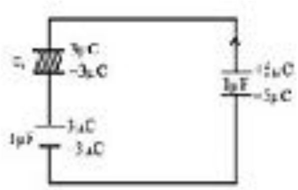
$$d = 2.09\ m$$

11. Three identical capacitors C_1, C_2 and C_3 have a capacitance of $1.0\ \mu F$ each and they are uncharged initially. They are connected in a circuit as shown in the figure and C_1 is then filled completely with a dielectric material of relative permittivity ϵ_r. The cell electromotive force (emf) $V0\ =\ 8\ V$. First the switch S_1 is closed while the switch S_2 is kept open. When the capacitor C_3 is fully charged, S_1 is opened and S_2 is closed simultaneously. When all the capacitors reach equilibrium, the charge on C_3 is found to be $5\ \mu C$. The value of ϵ_r =____________.



Solution: (1.50)



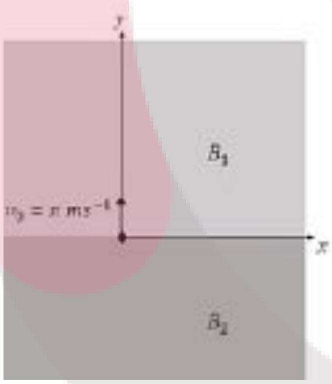


Applying loop rule

$$\frac{5}{1}-\frac{3}{\epsilon_r}-\frac{3}{1}=0 \qquad \frac{3}{\epsilon_r}=2$$

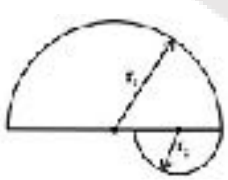
$$\epsilon_r=1.50$$

12. In the xy - plane, the region $y > 0$ has a uniform magnetic field $B_1\hat{k}$ and the region $y < 0$ has another uniform magnetic field $B_2\hat{k}$. A positively charged particle is projected from the origin along the positive y - axis with speed $v_0 = \pi\, m\, s^{-1}$ at $t = 0$, as shown in the figure. Neglect gravity in this problem. Let $t = T$ be the time when the particle crosses the x - axis from below for the first time. If $B_2 = 4B_1$, the average speed of the particle, in $m\, s^{-1}$, along the x - axis in the time interval T is __________



Solution: (2.00)

(i) Average speed along x – axis



$$\langle V_x \rangle = \frac{\int \left| \vec{v}_x \right| dt}{\int dt} = \frac{d_1 + d_2}{t_1 + t_2}$$

(ii) We have,

$$r_1 = \frac{mv}{qB_1}, r_2 = \frac{mv}{qB_2}$$

Since, $B_1 = \dfrac{B_2}{4}$

$\therefore \quad r_1 = 4r_2$

Time in $B_1 \Rightarrow \dfrac{\pi m}{qB_1} = t_1$

Time in $B_2 \Rightarrow \dfrac{\pi m}{qB_2} = t_2$

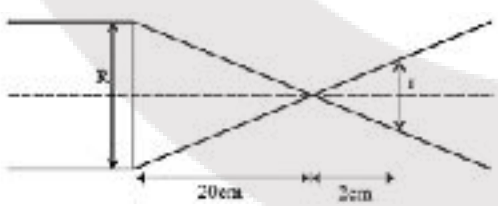
Total distance along x - axis $d_1 + d_2 = 2r_1 + 2r_2 = 2(r_1 + r_2) = 2(5r_2)$

Total time $T = t_1 + t_2 = 5t_2$

$\therefore \quad$ Average speed $= \dfrac{10r_2}{5t_2} = \dfrac{2mv}{qB_2} \times \dfrac{qB_2}{\pi m} = 2$

13. Sunlight of intensity $1.3\ kW\,m^{-2}$ is incident normally on a thin convex lens of focal length $20\ cm$. Ignore the energy loss of light due to the lens and assume that the lens aperture size is much smaller than its focal length. The average intensity of light, in $kW\,m^{-2}$, at a distance $22\ cm$ from the lens on the other side is __________.

Solution: (130)



$$\frac{r}{R} = \frac{2}{20} = \frac{1}{10}$$

$\therefore \quad$ Ratio of area $= \dfrac{1}{100}$

Let energy incident on lens be E

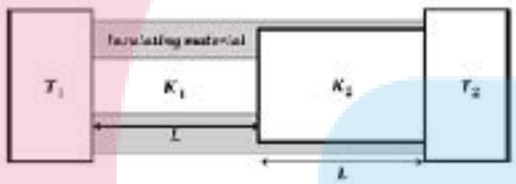
$\therefore$ Given $\dfrac{E}{A} = 1.3$

$E = A \times 1.30$

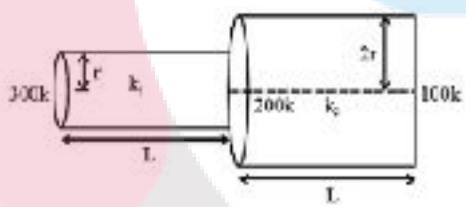
Also $\dfrac{a}{A} = \dfrac{1}{100}$

$\therefore$ Average intensity of light at $22\ cm = \dfrac{E}{a} = \dfrac{A \times 1.3}{a} = 100 \times 1.3 = 130k\ W/m^2$

14. Two conducting cylinders of equal length but different radii are connected in series between two heat baths kept at temperatures $T_1 = 300\ K$ and $T_2 = 100\ K$, as shown in the figure. The radius of the bigger cylinder is twice that of the smaller one and the thermal conductivities of the materials of the smaller and the larger cylinders are K_1 and K_2 respectively. If the temperature at the junction of the two cylinders in the steady state is $200\ K$, then K_1/K_2 =__________.



Solution: (4.00)



We have in steady state,

$$\left(\frac{200-300}{\frac{L}{k_1 \pi r^2}}\right) + \left(\frac{200-100}{\frac{L}{k_2 \pi (2r)^2}}\right) = 0$$

$$\Rightarrow \frac{k_1 \pi r^2 \times 100}{L} = \frac{100 k_2 \pi \times 4r^2}{L}$$

$$\Rightarrow \frac{k_1}{k_2} = 4$$

Linked Comprehension Type

PARAGRAPH "X"

15. In electromagnetic theory, the electric and magnetic phenomena are related to each other. Therefore, the dimensions of electric and magnetic quantities must also be related to each other. In the questions below, $[E]$ and $[B]$ stand for dimensions of electric and magnetic fields respectively, while $[\epsilon_0]\ and [\mu_0]$ stand for dimensions of the permittivity and permeability of free space respectively. $[L]$and $[T]$ are dimensions of length and time respectively. All the quantities are given in SI units

The relation between $[E]\ and\ [B]$ is

(A) $[E]=[B][L][T]$

(B) $[E]=[B][L]^{-1}[T]$

(C) $[E]=[B][L][T]^{-1}$

(D) $[E]=[B][L]^{-1}[T]^{-1}$

Solution: (C)

We have $\frac{E}{C}=B$

$$\therefore\ [B]=\frac{[E]}{[C]}=[E]L^{-1}T^{1}$$

$$\Rightarrow\ [E]=[B][L][T]^{-1}$$

PARAGRAPH "X"

16. In electromagnetic theory, the electric and magnetic phenomena are related to each other. Therefore, the dimensions of electric and magnetic quantities must also be related to each other. In the questions below, $[E]$ and $[B]$ stand for dimensions of electric and magnetic fields respectively, while $[\epsilon_0]$ and $[\mu_0]$ stand for dimensions of the permittivity and permeability of free space respectively. $[L]$and $[T]$are dimensions of length and time respectively. All the quantities are given in SI units

The relation between $[\epsilon_0]$ and $[\mu_0]$ is

(A) $[\mu_0]=[\epsilon_0][L]^2[T]^{-2}$

(B) $[\mu_0]=[\epsilon_0][L]^{-2}[T]^2$

(C) $[\mu_0] = [\epsilon_0]^{-1}[L]^2[T]^{-2}$

(D) $[\mu_0] = [\epsilon_0]^{-1}[L]^{-2}[T]^2$

Solution: (D)

We have,

$$C = \frac{1}{\sqrt{\mu_0 \in_0}}$$

$$\therefore \quad [C^2] = \left[\frac{1}{\mu_0 \in_0}\right]$$

$$\Rightarrow \quad L^2T^{-2} = \frac{1}{[\mu_0][\in_0]}$$

$$\Rightarrow \quad [\mu_0] = [\epsilon_0]^{-1}[L]^{-2}[T]^2$$

PARAGRAPH "A"

17. If the measurement errors in all the independent quantities are known, then it is possible to determine the error in any dependent quantity. This is done by the use of series expansion and truncating the expansion at the first power of the error. For example, consider the relation $z = \frac{x}{y}$. If the errors in $x,\ y$ and z are $\Delta x,\ \Delta y$ and Δz, respectively, then

$$z \pm \Delta z = \frac{x \pm \Delta x}{y \pm \Delta y} = \frac{x}{y}\left(1 \pm \frac{\Delta x}{x}\right)\left(1 \pm \frac{\Delta y}{y}\right)^{-1}$$

The series expansion for $\left(1 \pm \frac{\Delta y}{y}\right)^{-1}$, to first power in $\Delta y/y$, is $1 \mp (\Delta y/y)$. The relative errors in independent variables are always added. So the error in z will be $\Delta z = z(\Delta x/x + \Delta y/y)$. The above derivation makes the assumption that $\mathbf{\Delta} x/x \ll 1, \Delta y/y \ll 1$. Therefore, the higher powers of these quantities are neglected

Consider the ratio $r = \frac{(1-a)}{(1+a)}$ to be determined by measuring a dimensionless quantity a. If the error in the measurement of a is $\Delta a\ (\Delta a/a \ll 1)$, then what is the error Δr in determining r?

(A) $\frac{\Delta a}{(1+a)^2}$

(B) $\frac{2\Delta a}{(1+a)^2}$

(C) $\frac{2\Delta a}{(1-a)^2}$

(D) $\frac{2a\Delta a}{(1-a)^2}$

Solution: (B)

$$r = \left(\frac{1-a}{1+a}\right)$$

$$\frac{\Delta r}{r} = \frac{\Delta(1-a)}{(1-a)} + \frac{\Delta(1+a)}{(1+a)}$$

$$= \frac{\Delta a}{(1-a)} + \frac{\Delta a}{(1+a)}$$

$$= \frac{\Delta a(1+a+1-a)}{(1-a)(1+a)}$$

$$\therefore \quad \Delta r = \frac{2\Delta a}{(1-a)(1+a)} \frac{(1-a)}{(1+a)} = \frac{2\Delta a}{(1+a)^2}$$

PARAGRAPH "A"

18. If the measurement errors in all the independent quantities are known, then it is possible to determine the error in any dependent quantity. This is done by the use of series expansion and truncating the expansion at the first power of the error. For example, consider the relation $z = \frac{x}{y}$. If the errors in $x,\ y$ and z are $\Delta x,\ \Delta y$ and Δz, respectively, then

$$z \pm \Delta z = \frac{x \pm \Delta x}{x \pm \Delta y} = \frac{x}{y}\left(1 \pm \frac{\Delta x}{x}\right)\left(1 \pm \frac{\Delta y}{y}\right)^{-1}$$

The series expansion for $\left(1 \pm \frac{\Delta y}{y}\right)^{-1}$, to first power in $\Delta y/y$, is $1 \mp (\Delta y/y)$. The relative errors in independent variables are always added. So the error in z will be $\Delta z = z(\Delta x/x + \Delta y/y)$. The above derivation makes the assumption that $\Delta x/x \ll 1, \Delta y/y \ll 1$. Therefore, the higher powers of these quantities are neglected

In an experiment the initial number of radioactive nuclei is 3000. It is found that 1000 ± 40 nuclei decayed in the first $1.0\ s$. For $|x| \ll 1, \ln(1+x) = x$ up to first power in x. The error $\Delta\lambda$, in the determination of the decay constant λ, in s^{-1}, is

(A) 0.04

(B) 0.03

(C) 0.02

(D) 0.01

Solution: (C)

$N = N_0 e^{-\lambda t}$

$\ln N = \ln N_0 - \lambda t$

$\frac{dN}{N} = -d\lambda t$

Converting to error,

$\frac{\Delta N}{N} = \Delta\lambda t$

$\therefore \quad \Delta\lambda = \frac{40}{2000 \times L} = 0.02$ (N is number of nuclei left undecayed)

CHEMISTRY

Multiple Correct Answer Type

1. The compound(s) which generate(s) N_2 gas upon thermal decomposition below 300^oC is (are)

(A) NH_4NO_3

(B) $(NH_4)_2 Cr_2O_7$

(C) $Ba(N_3)_2$

(D) Mg_3N_2

Solution: (B, C)

(A)

$$NH_4NO_3 \xrightarrow[\text{below } 300^\circ C]{\Delta} N_2O + 2H_2O$$

$(B)\ \left(NH_4\right)_2Cr_2O_7 \xrightarrow{\Delta} N_2 + Cr_2O_3 + 4H_2O$

$(C)\ Ba\left(N_3\right)_2 \xrightarrow{\Delta} Ba + 3N_2$

$(D)\ Mg_3N_2$ (It does not decompose into N_2)

2. The correct statement(s) regarding the binary transition metal carbonyl compounds is (are) (Atomic numbers: $Fe = 26, Ni = 28$)

(A) Total number of valence shell electrons at metal center in $Fe(CO)_5$ or $Ni(CO)_4$ is 16

(B) These are predominantly low spin in nature

(C) Metal – carbon bond strengthens when the oxidation state of the metal is lowered

(D) The carbonyl $C-O$ bond weakens when the oxidation state of the metal is increased

Solution: (B, C)

$(A)\ \left[Fe\left(CO_5\right)\right]$ and $\left(Ni(CO)_4\right)$ Complexes have 18- electrons in their valence shell

(B) Carbonyl complexes are predominantly low spin complexes due to strong ligand field

(C) As electron density increases on metals (with lowering oxidation state on metals), the extent of synergic bonding increases. Hence M – C bond strength increases

(D) While positive charge on metals increases and the extent of synergic bond decreases and hence C – O bond becomes stronger.

3. Based on the compounds of group 15 elements, the correct statement(s) is (are)

(A) Bi_2O_5 is more basic than N_2O_5

(B) NF_3 is more covalent than BiF_3

(C) PH_3 Boils at lower temperature than NH_3

(D) The $N-N$ single bond is stronger than the $P-P$ single bond

Solution: (A, B, C)

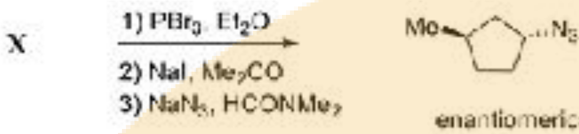
$(A)\ Bi_2O_5$ is metallic oxide but N_2O_5 is non metallic oxide therefore Bi_2O_5 is basic but N_2O_5 is acidic.

(B) In NF_3 , N and F are non metals but BiF_3 , Bi is metal but F is non metal therefore NF_3 is more covalent than BiF_3

(C) In PH_3 hydrogen bonding is absent but in NH_3 hydrogen bonding is present therefore PH_3 boils at lower temperature than NH_3

(D) Due to small size in N-N single bond l.p. – l.p. repulsion is more than P-P single bond therefore N-N single bond is weaker than the P-P single bond

4. In the following reaction sequence, the correct structure(s) of X is (are)

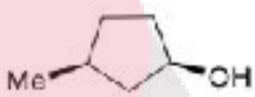


(A)

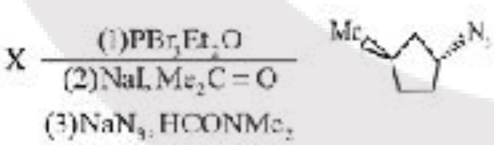
(B)

(C)

(D)



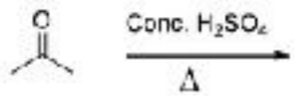
Solution: (B)



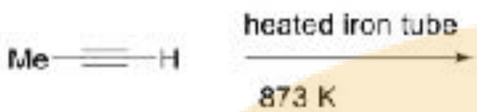
All the three reaction are S_{N^2} so X is

5. The reaction(s) leading to the formation of 1, 3, 5-trimethylbenzene is (are)

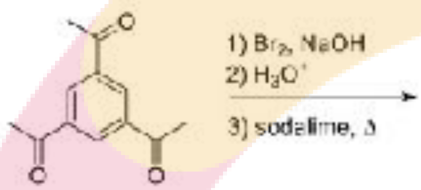
(A)



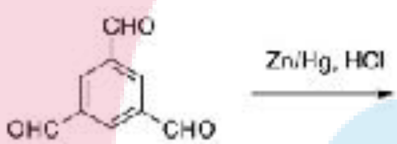
(B)



(C)

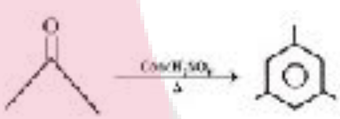


(D)

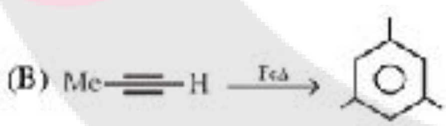


Solution: (A, B, D)

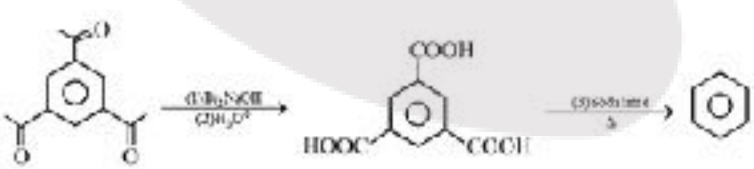
(A)



(B)

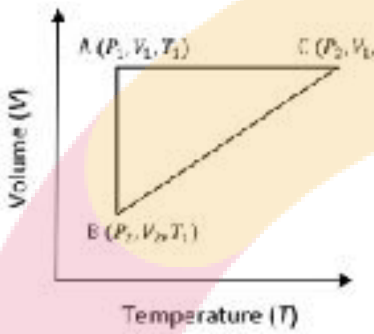


(C)



(D)

6. A reversible cyclic process for an ideal gas is shown below. Here, $P,\ V,$ and T are pressure, volume and temperature, respectively. The thermodynamic parameters $q,\ w,\ H$ and U are heat, work, enthalpy and internal energy, respectively.



The correct option(s) is (are)

(A) $q_{AC} = \Delta U_{BC}$ and $w_{AB} = P_2(V_2 - V_1)$

(B) $w_{BC} = P_2(V_2 - V_1)$ and $q_{BC} = \Delta H_{AC}$

(C) $\Delta H_{CA} < \Delta U_{CA}$ and $q_{AC} = \Delta U_{BC}$

(D) $q_{BC} = \Delta H_{AC}$ and $\Delta H_{CA} > \Delta U_{CA}$

Solution: (B, C)

AC $\rightarrow$ Isochoric

AB $\rightarrow$ Isothermal

BC $\rightarrow$ Isobaric

$\#q_{AC} = \Delta U_{BC} = nC_V(T_2 - T_1)$

$W_{AB} = nRT_1 \ln\left(\frac{V_2}{V_1}\right)$ A (wrong)

$\#q_{BC} = \Delta H_{AC} = nC_P(T_2 - T_1)$

$W_{BC} = -P_2(V_1 - V_2)$ B (correct)

$\#nC_P(T_1 - T_2) < nC_V(T_1 - T_2)$ C (correct)

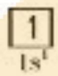
$\Delta H_{CA} < \Delta U_{CA}$

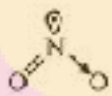
#D (wrong)

7. Among the species given below, the total number of diamagnetic species is ___.

H atom, NO_2 monomer, O_2^- (superoxide), dimeric sulphur in vapour phase, Mn_3O_4, $(NH_4)_2[FeCl_4]$, $(NH_4)_2[NiCl_4]$, K_2MnO_4, K_2CrO_4

Solution: (1)

H - atom =  paramagnetic

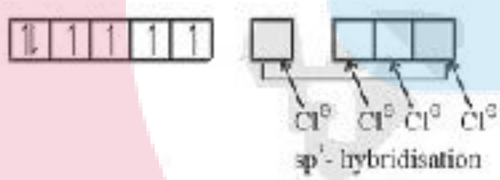
$NO_2 =$  odd unpaired electron species paramagnetic

O_2^-(superocide) = One unpaired electrons inπ^* M.O paramagnetic

S_2 (in vapour phase) = same as O_2 , two unpaired $e^- s$ are present in π^* M.O paramagnetic

$Mn_3O_4 = 2M^{+2}nO \,.\, M^{+4}nO_2$ paramagnetic

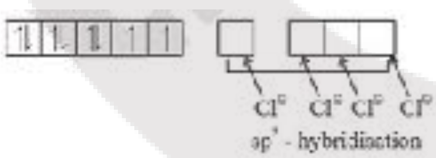
$(NH_4)_2[FeCl_4] = Fe^{+2} = 3d^6\ 4s^0$



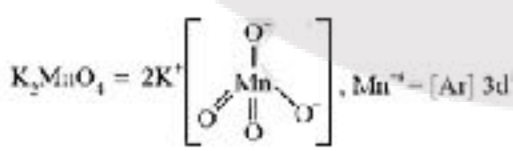
paramagnetic

$(NH_4)_2[NiCl_4] = Ni = 3d^8\ 4s^2$

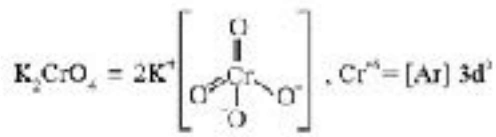
$Ni^{+2} = 3d^8\ 4s^0$



paramagnetic



paramagnetic



diamagnetic

8. The ammonia prepared by treating ammonium sulphate with calcium hydroxide is completely used by $NiCl_2 . 6H_2O$ to form a stable coordination compound. Assume that both the reactions are 100% complete. If 1584 g of ammonium sulphate and 952 g of $NiCl_2 . 6H_2O$ are used in the preparation, the combined weight (in grams) of gypsum and the nickelammonia coordination compound thus produced is ____. (Atomic weights in $g\ mol^{-1}: H = 1, N = 14, O = 16, S = 32, Cl = 35.5, Ca = 40, Ni = 59$)

Solution: (2992)

$$\underset{\substack{1584g \\ =12\ mol}}{(NH_4)_2SO_4} + Ca(OH)_2 \rightarrow \underset{\substack{gypsum\ (M=172) \\ 12\ mol}}{CaSO_4.2H_2O} + \underset{24\ mole}{2NH_3}$$

$$\underset{952\ g=4\ mol}{NiCl_2 \cdot 6H_2O} + \underset{24\ mol}{6NH_3} \rightarrow \underset{\substack{(M=232) \\ 4\ mol}}{[Ni(NH_3)_6]Cl_2} + 6H_2O$$

Total mass $= 12 \times 172 + 4 \times 232 = 2992\ g$

9. Consider an ionic solid MX with $NaCl$ structure. Construct a new structure (Z) whose unit cell is constructed from the unit cell of MX following the sequential instructions given below. Neglect the charge balance.

(i) Remove all the anions (X) except the central one

(ii) Replace all the face centered cations (M) by anions (X)

(iii) Remove all the corner cations (M)

(iv) Replace the central anion (X) with cation (M)

The value of $\left(\frac{\text{number of anions}}{\text{number of cations}}\right)$ in Z is _____.

Solution: (3)

$X^- \Rightarrow O.V$

$M^+ \Rightarrow FCC$

	M^+	X^-
(i)	4	1
(ii)	$4-3$	$3+1$
(iii)	$4-3-1$	$3+1$
(iv)	1	3

$Z = \frac{3}{1} = 3$

10. For the electrochemical cell, $Mg(s)\left|Mg^{2+}(aq, 1\ M)\right|\left|Cu^{2+}(aq, 1\ M)\right|Cu\ (s)$ the standard emf of the cell is $2.70\ V\ at\ 300\ K$. When the concentration of Mg^{2+} is changed to $x\ M$, the cell potential changes to $2.67\ V\ at\ 300\ K$. The value of x is ____.

(given, $\frac{F}{R} = 11500\ K\ V^{-1}$, where F is the Faraday constant and R is the gas constant, $\ln(10) = 2.30$)

Solution: (10)

$Mg(s) + Cu^{2+}(aq) \rightarrow Mg^{2+}(aq) + Cu(s)$

$E^o_{cell} = 2.70\ \ E_{cell} = 2.67\ \ \ Mg^{2+} = x\ M\ \ Cu^{2+} = 1\ M$

$E_{cell} = E^o_{cell} - \frac{RT}{nF}\ln x$

$2.67 = 2.70 - \frac{RT}{2F}\ln x$

$-0.03 = -\frac{R \times 300}{2F} \times \ln x$

$\ln x = \frac{0.03 \times 2}{300} \times \frac{F}{R}$

$= \frac{0.03 \times 2 \times 11500}{300 \times 1}$

$$\ln x = 2.30 = \ln(10)$$

$$x = 10$$

11. A closed tank has two compartments **A** and **B**, both filled with oxygen (assumed to be ideal gas). The partition separating the two compartments is fixed and is a perfect heat insulator (Figure 1). If the old partition is replaced by a new partition which can slide and conduct heat but does NOT allow the gas to leak across (Figure 2), the volume (in m^3) of the compartment A after the system attains equilibrium is ____.

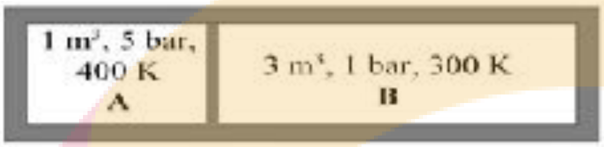


Figure 1

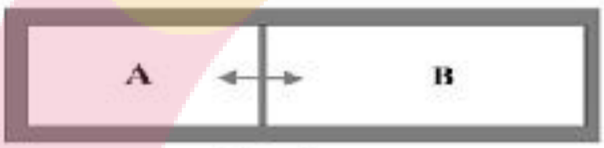


Figure 2

Solution: (2.22)

$$P_1 = 5$$

$$v_1 = 1$$

$$T_1 = 400$$

$$n_1 = \frac{5}{400R}$$

$$P_2 = 1$$

$$v_2 = 3$$

$$T_2 = 300$$

$$n_2 = \frac{3}{300\,R}$$

Let volume be $(v + x)$ $\quad v = (3 - x) \quad 15 - 5x = 4 + 4x$

$$\frac{P_A}{T_A} = \frac{P_B}{T_B}$$

$$\Rightarrow \frac{n_{b_1} \times R}{v_{b_1}} = \frac{n_{b_2} \times R}{v_{b_2}}$$

$$\Rightarrow \frac{5}{400(4+x)} = \frac{3}{300R(3-x)}$$

$$\Rightarrow 5(3-x) = 4 + 4x$$

$$\Rightarrow x = \frac{11}{9}$$

$$v = 1 + x = 1 + \frac{11}{9} = \left(\frac{20}{9}\right) = 2.22$$

12. Liquids A and B form ideal solution over the entire range of composition. At temperature T, equimolar binary solution of liquids A and B has vapour pressure 45 Torr. At the same temperature, a new solution of A and B having mole fractions x_A and x_B, respectively, has vapour pressure of 22.5 Torr. The value of $\frac{x_A}{x_B}$ in the new solution is ____.

(Given that the vapour pressure of pure liquid A is 20 Torr at temperature T)

Solution: (19)

$$45 = P_A^o \times \frac{1}{2} + P_B^o \times \frac{1}{2}$$

$P_A^o + P_B^o = 90$ .... (i)

Given $P_A^o = 20\ torr$

$P_B^o = 70\ torr$

$$\Rightarrow 22.5\ torr = 20x_A + 70(1 - x_A)$$

$$= 70 - 50\ x_A$$

$$x_A = \left(\frac{70 - 22.5}{50}\right) = 0.95$$

$x_B = 0.05$

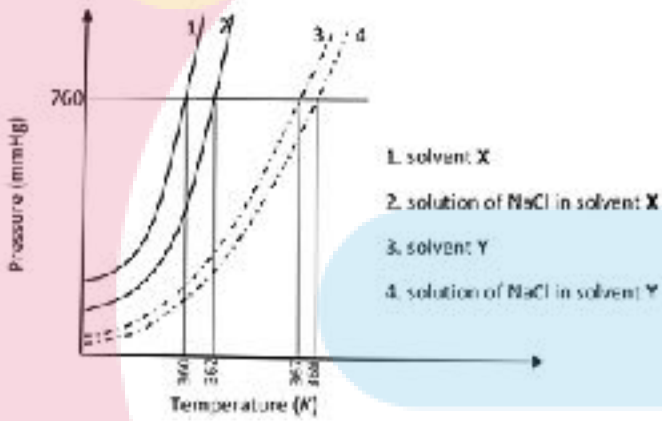
So $\frac{X_A}{X_B} = \frac{0.95}{0.05} = 19$

13. The solubility of a salt of weak acid (AB) at $pH\,3$ is $Y \times 10^{-3}\,mol\,L^{-1}$. The value of Y is ____. (Given that the value of solubility product of $AB\left(K_{sp}\right) = 2 \times 10^{-10}$ and the value of ionization constant of $HB(Ka) = 1 \times 10^{-8}$)

Solution: (4.47)

$$S = \sqrt{K_{sp}\left(\frac{[H^+]}{K_a} + 1\right)} = \sqrt{2 \times 10^{-10}\left(\frac{10^{-3}}{10^{-8}} + 1\right)} \simeq \sqrt{2 \times 10^{-5}} = 4.47 \times 10^{-3} M$$

14. The plot given below shows $P - T$ curves (where P is the pressure and T is the temperature) for two solvents X and Y and isomolal solutions of $NaCl$ in these solvents. $NaCl$ Completely dissociates in both the solvents.



On addition of equal number of moles of a non - volatile solute S in equal amount (in kg) of these solvents, the elevation of boiling point of solvent X is three times that of solvent Y. Solute S is known to undergo dimerization in these solvents. If the degree of dimerization is 0.7 in solvent Y, the degree of dimerization in solvent X is ____.

Solution: (0.05)

From graph

For solvent X' $\quad \Delta T_{bx} = 2$

$\Delta T_{bx} = m_{NaCl} \times K_{b(x)}$ .... (i)

For solvent 'Y' $\quad \Delta T_{by} = 1$

$\Delta T_{b(y)} = m_{NaCl} \times K_{b(y)}$ .... (ii)

Equation (i)/(ii)

$$\Rightarrow \frac{K_{b(x)}}{K_{b(y)}} = 2$$

For solute S

$$\underset{\substack{1\\1-\alpha}}{2(S)} \rightarrow \underset{\alpha/2}{S_2}$$

$$i = \left(1 - \frac{\alpha}{2}\right)$$

$$\Delta T_{b(x)(s)} = \left(1 - \frac{\alpha_1}{2}\right) K_{b(x)}$$

$$\Delta T_{b(y)(s)} = \left(1 - \frac{\alpha_2}{2}\right) K_{b(y)}$$

Given $\Delta T_{b(x)(s)} = 3\,\Delta T_{b(y)(s)}$

$$\left(1 - \frac{\alpha_1}{2}\right) K_{b(x)} = 3 \times \left(1 - \frac{\alpha_1}{2}\right) \times k_{b(y)}$$

$$2\left(1 - \frac{\alpha_1}{2}\right) = 3\left(1 - \frac{\alpha_2}{2}\right)$$

$$\alpha_2 = 0.7$$

So $\alpha = 0.05$

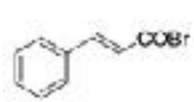
Paragraph "X"

15. Treatment of benzene with CO/HCl in the presence of anhydrous $AlCl_3/CuCl$ followed by reaction with $Ac_2O/NaOAc$ gives compound X as the major product. Compound X upon reaction with

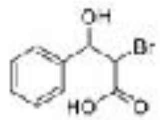
Br_2/Na_2CO_3, followed by heating at 473 K with moist KOH furnishes Y as the major product. Reaction of X with $H_2/Pd-C$, followed by H_3PO_4 treatment gives Z as the major product

The compound Y is

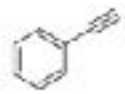
(A)



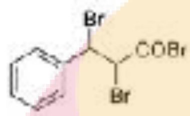
(B)



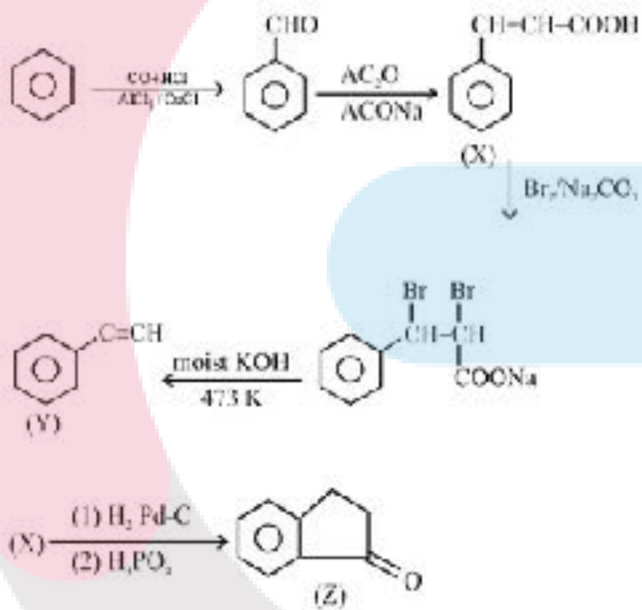
(C)



(D)



Solution: (C)



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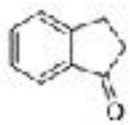
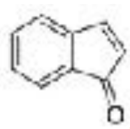
Br_2/Na_2CO_3, followed by heating at 473 K with moist KOH furnishes Y as the major product. Reaction of X with $H_2/Pd - C$, followed by H_3PO_4 treatment gives Z as the major product

The compound Z is

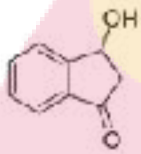
(A)



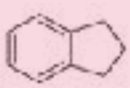
(B)



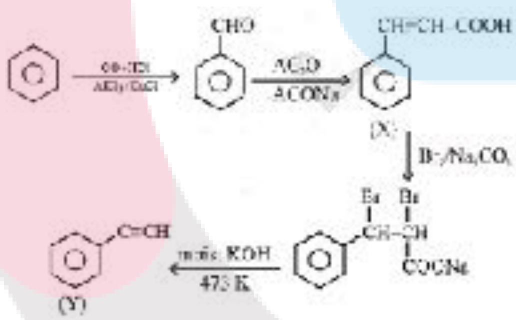
(C)



(D)

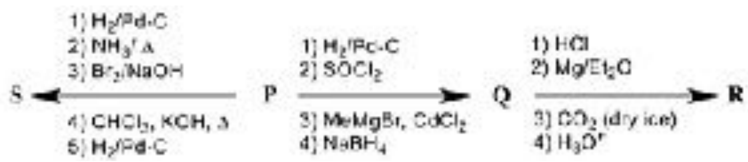


Solution: (A)



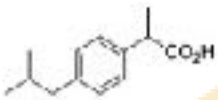
Paragraph "A"

17. An organic acid $P\ (C_{11}H_{12}O_2)$ can easily be oxidized to a dibasic acid which reacts with ethyleneglycol to produce a polymer dacron. Upon ozonolysis, P gives an aliphatic ketone as one of the products. P undergoes the following reaction sequences to furnish R via Q. The compound P also undergoes another set of reactions to produce S.

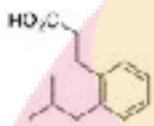


The compound R is

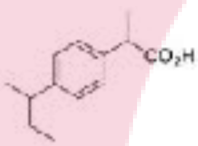
(A)



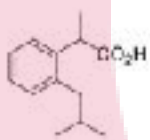
(B)



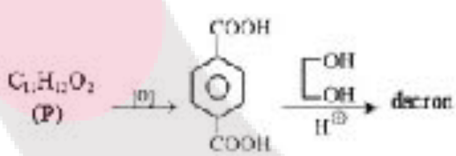
(C)

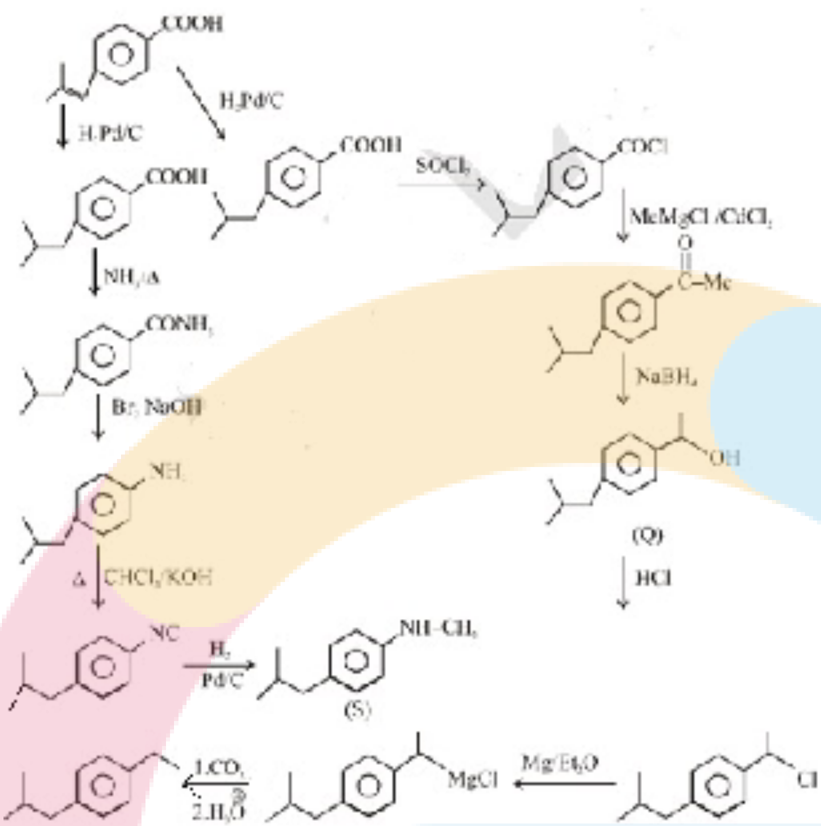


(D)

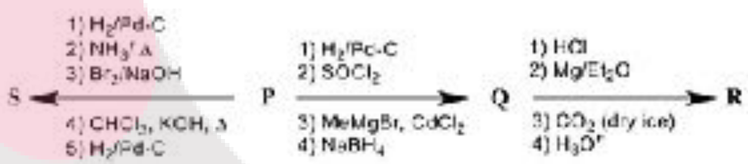


Solution: (A)



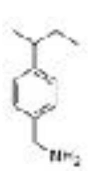


18. An organic acid $P\ (C_{11}H_{12}O_2)$ can easily be oxidized to a dibasic acid which reacts with ethyleneglycol to produce a polymer dacron. Upon ozonolysis, P gives an aliphatic ketone as one of the products. P undergoes the following reaction sequences to furnish R via Q. The compound P also undergoes another set of reactions to produce S.

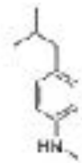


The compound S is

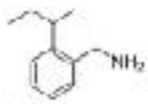
(A)



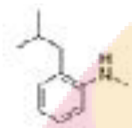
(B)



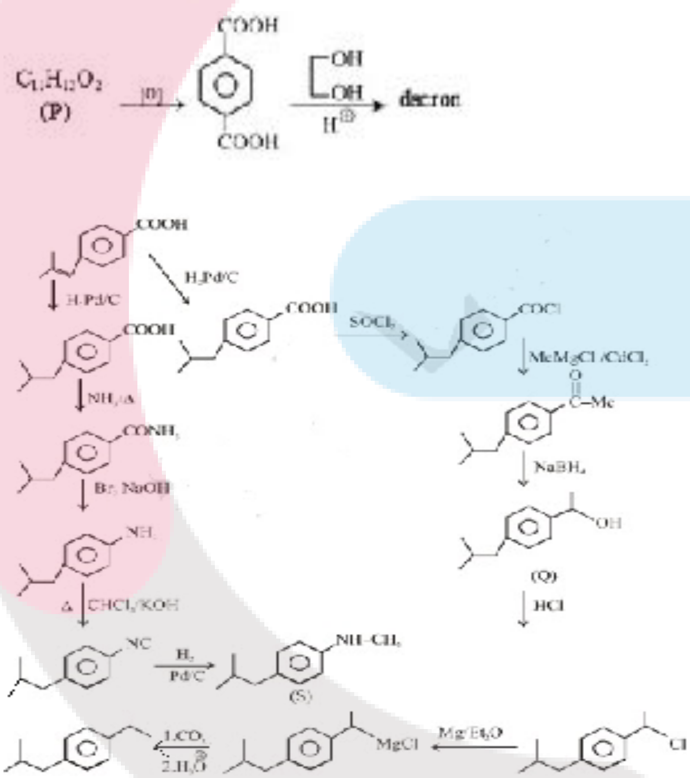
(C)



(D)



Solution: (B)



MATHS

Multiple Correct Answer Type

1. For a non-zero complex number z, let average (z) denote the principal argument with $-\pi < \text{average}\ (z) \leq \pi$ Then, which of the following statement(s) is (are) FALSE?

(A) Average $(-1-i)=\frac{\pi}{4}$, where $i=\sqrt{-1}$

(B) The function $f:\mathbb{R}\rightarrow(-\pi,\pi]$ defined by $f(t)=\text{average}(-1+\text{it})$ for all $t\in\mathbb{R}$, is continuous at all points of $\mathbb{R}$, where $i=\sqrt{-1}$

(C) For any two non-zero comple x numbers z_1 and z_2, $\arg\left(\frac{z_1}{z_2}\right)-\arg(z_1)+\arg(z_2)$ is an integer multiple of 2π

(D) For any three given distinct complex numbers z_1, z_2 and z_3, the locus of the point z satisfying the condition $average\left(\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}\right)=\pi$ lies on a straight line

Solution: (A, B, D)

$$(A)\ \text{Average}=(-1-i)=-\frac{3\pi}{4}$$

$$(B)\ \text{average}(-1+\text{it})=\begin{cases}\pi-\tan^{-1}(t), & t\geq 0\\ -\pi+\tan^{-1}(t), & t<0\end{cases}$$

Discontinuous at $t=0$

$$(C)\ \arg\left(\frac{z_1}{z_2}\right)-\arg(z_1)+\arg(z_2)$$

$$=\ \arg z_1-\arg(z_2)+2n\pi-\arg(z_1)+\arg(z_2)=2n\pi$$

$$(D)\ arg\left(\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}\right)=\pi$$

$\Rightarrow\ \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$ is real

$\Rightarrow\ z,\ z_1,\ z_2,\ z_3$ are concyclic

2. In a triangle PQR, $let\ \angle PQR=30^o$ and the sides PQ and QR have lengths $10\sqrt{3}$ and 10, respectively. Then, which of the following statement(s) is (are) TRUE?

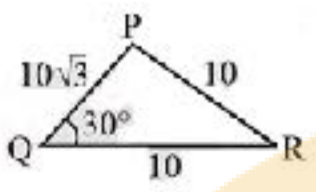
(A) $\angle PQR=45^o$

(B) The area of the triangle PQR is $25\sqrt{3}$ and $\angle PQR=120^o$

(C) The radius of the in circle of the triangle PQR is $10\sqrt{3}-15$

(D) The area of the circumcircle of the triangle PQR is 100π

Solution: (B, C, D)



$$\cos 30^o = \frac{\left(10\sqrt{3}\right)^2 + (10)^2 - (PR)^2}{2\times 10\sqrt{3}\times 10}$$

$$\Rightarrow \quad PR = 10$$

$$\because \quad QR = PR \Rightarrow \angle PQR = \angle QPR$$

$$\angle QPR = 30^o$$

(B) Area of $\Delta PQR = \frac{1}{2}\times 10\sqrt{3}\times 10\times \sin 30^o = \frac{1}{2}\times 10\times 10\sqrt{3}\times\frac{1}{2}$

$$= \ 25\sqrt{3}$$

$$\angle QRP = 180^o - \left(30^o + 30^o\right) = 120^o$$

$$(C)\ r = \frac{\Delta}{S} = \frac{25\sqrt{3}}{\left(\frac{10+10+10\sqrt{3}}{2}\right)} = \frac{25\sqrt{3}}{10+5\sqrt{3}}$$

$$= \ 5\sqrt{3}\,.\left(2-\sqrt{3}\right) = 10\sqrt{3}-15$$

$$(D)\ R = \frac{a}{2\sin A} = \frac{10}{2\sin 30^o} = 10$$

$$\therefore \quad \text{Area} = \pi R^2 = 100\pi$$

3. Let $P_1: 2x + y - z = 3$ and $P_2: x + 2y + z = 2$ be two planes. Then, which of the following statement(s) is (are) TRUE?

(A) The line of intersection of P_1 and P_2 has direction ratios $1, 2, -1$

(B) The line $\frac{3x-4}{9}=\frac{1-3y}{9}=\frac{z}{3}$ is perpendicular to the line of intersection of P_1 and P_2

(C) The acute angle between P_1 and P_2 is 60^o

(D) If P_3 is the plane passing through the point $(4, 2, -2)$ and perpendicular to the line of intersection of P_1 and P_2, then the distance of the point $(2,1,1)$ from the plane P_2 is $\frac{2}{\sqrt{3}}$

Solution: (C, D)

D.C of line intersection $(a,\ b,\ c)$

$\Rightarrow\ \ 2a+b-c=0$

$a+2b+c=0$

$\frac{a}{1+2}=\frac{b}{-1-2}=\frac{c}{4-1}$

$\therefore D.C$ is $(1,\ -1,\ 1)$

$(B)\ \frac{3x-4}{9}=\frac{1-3y}{9}=\frac{z}{3}$

$\Rightarrow\ \frac{x-\frac{4}{3}}{3}=\frac{y-\frac{1}{3}}{-3}=\frac{z}{3}$

$\Rightarrow$ lines are parallel

(C) Acute angle between P_1 and $P_2=\cos^{-1}\left(\frac{2\times1+1\times2-1\times1}{\sqrt{6}\sqrt{6}}\right)$

$=\ \cos^{-1}\left(\frac{3}{6}\right)=\cos^{-1}\left(\frac{1}{2}\right)=60^o$

(D) Plane is given by $(x-4)-(y-2)+(z+2)=0$

$\Rightarrow\ \ x-y+z=0$

Distance of (2, 1, 1) from plane $=\frac{2-1+1}{\sqrt{3}}=\frac{2}{\sqrt{3}}$

4. For every twice differentiable function $f:\mathbb{R} \to [-2, 2]$ with $(f(0))^2 + (f'(0))^2 = 85$, which of the following statement(s) is (are) TRUE?

(A) There exist $r, s \in \mathbb{R}$, where $r < s$, such that f is one-one on the open interval (r, s)

(B) There exists $x_0 \in (-4,0)$ such that $\left|f'(x_0)\right| \leq 1$

(C) $\lim_{x\to\infty} f(x) = 1$

(D) There exists $\alpha \in (-4, 4)$ such that $f(\alpha) + f''(\alpha) = 0$ and $f'(\alpha) \neq 0$

Solution: (A, B, D)

$f(x)$ can't be constant throughout the domain. Hence we can find $x \in (r, s)$ such that $f(x)$ is one-one option (A) is true.

$(B)\ \left|f'(x)\right| = \left|\frac{f(0) - f(-4)}{4}\right| \leq 1\ (LMVT)$

$(C)\ f(x) = \sin\left(\sqrt{85x}\right)$ satisfies given condition

But $\lim_{x\to\infty} \sin\left(\sqrt{85}\right)\ D.N.E$

$\Rightarrow$ Incorrect

$(D)\ g(x) = f^2(x) + \left(f'(x)\right)^2$

$\left|f'(x_1)\right| \leq 1$ (by LMVT)

$\left|f(x_1)\right| \leq 2$ (given)

$\Rightarrow\ g(x_1) \leq 5\ \ \exists x_1 \in (-4,0)$

Similarly $g(x_2) \leq 5\ \ \exists x_2 \in (0,4)$

$g(0) = 85 \Rightarrow g(x)$ has maxima in (x_1, x_2) say at α

$g'(\alpha) = 0$ and $g(\alpha) \geq 85$

$2f'(\alpha)\left(f(\alpha) + f''(\alpha)\right) = 0$

If $f'(\alpha) = 0 \Rightarrow g(\alpha) = f^2(\alpha) \geq 85$ not possible

$\Rightarrow\ f(\alpha) + f''(\alpha) = 0\ \exists \alpha \in (x_1, x_2) \in (-4,4)$

(D) correct

5. Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be two non-constant differentiable functions. If $f'(x)=\left(e^{(f(x))-g(x)}\right)g'(x)$ for all $x\in\mathbb{R}$, and $f(1)=g(2)=1$ then which of the following statement(s) is (are) TRUE?

(A) $f(2)<1-\log_e 2$

(B) $f(2)>1-\log_e 2$

(C) $g(1)>1-\log_e 2$

(D) $g(1)<1-\log_e 2$

Solution: (B, C)

$f'(x)=e^{(f(x)-g(x))}g'(x)\forall x\in\mathbb{R}$

$\Rightarrow\quad e^{-f(x)}.f'(x)-e^{-g(x)}g'(x)=0$

$\Rightarrow\quad \int\left(e^{-f(x)}f'(x)-e^{-g(x)}.g'(x)\right)dx=C$

$\Rightarrow\quad -e^{-f(x)+e^{-g(x)}}=C$

$\Rightarrow\quad -e^{-f(1)}+e^{-g(1)}=-e^{-f(2)}+e^{-g(2)}$

$\Rightarrow\quad -\frac{1}{e}+e^{g(1)}=-e^{-f(2)}+\frac{1}{e}$

$\Rightarrow\quad e^{-f(2)}+e^{-g(1)}=\frac{2}{e}$

$\therefore\quad e^{-f(2)}<\frac{2}{e}\text{and } e^{-g(1)}<\frac{2}{e}$

$\Rightarrow\quad -f(2)<\ln 2-1 \text{ and } -g(1)<\ln 2-1$

$\Rightarrow\quad f(2)>1-\ln 2 \text{ and } g(1)>1-\ln 2$

6. Let $f:[0,\infty)\to\mathbb{R}$ be a continuous function such that

$$f(x)=1-2x+\int_0^x e^{x-t}f(t)dt$$

For all $x \in [0,\infty)$. Then, which of the following statement(s) is (are) TRUE?

(A) The curve $y = f(x)$ passes through the point $(1, 2)$

(B) The curve $y = f(x)$ passes through the point $(2, -1)$

(C) The area of the region $\left\{(x,y) \in [0,1] \times \mathbb{R} : f(x) \le y \le \sqrt{1-x^2}\right\}$ $is \dfrac{\pi-2}{4}$

(D) The area of the region $\left\{(x,y) \in [0,1] \times \mathbb{R} : f(x) \le y \le \sqrt{1-x^2}\right\}$ $is \dfrac{\pi-1}{4}$

Solution: (B, C)

$$f(x) = 1 - 2x + \int_0^x e^{x-t} f(t)dt$$

$$\Rightarrow \quad e^{-x}f(x) = e^{-x}(1-2x) + \int_0^x e^{-t} f(t)dt$$

Differentiate w. r.t.x

$$e^{-x}f(x) + e^{+x}f'(x) = -e^{-x}(1-2x) + e^{-x}(-2) + e^{-x}f(x)$$

$$\Rightarrow \quad -f(x) + f'(x) = -(1-2x) - 2 + f(x)$$

$$\Rightarrow \quad f'(x) - 2f(x) = 2x - 3$$

Integrating factor $= e^{-2x}$

$$f(x).e^{-2x} = \int e^{-2x}(2x-3)dx$$

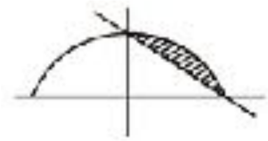
$$= \quad (2x-3)\int e^{-2x}dx - \int\left((2)\int e^{-2x}dx\right)dx$$

$$= \frac{(2x-3)e^{-2x}}{-2} - \frac{e^{-2x}}{2} + c$$

$$f(x) = \frac{2x-3}{-2} - \frac{1}{2} + ce^{2x}$$

$$f(0) = \frac{3}{2} - \frac{1}{2} + c = 1 \Rightarrow c = 0$$

$$\therefore \quad f(x) = 1 - x$$



Area $= \frac{\pi}{4} - \frac{1}{2} = \frac{\pi - 2}{4}$

7. The value of $\left((\log_2 9)^2\right)^{\frac{1}{\log_2(\log_2 9)}} \times \left(\sqrt{7}\right)^{\frac{1}{\log_4 7}}$ is_____.

Solution: (8)

$\log_2 9^{\frac{2}{\log_2(\log_2 9)}} \times 7^{\frac{\frac{1}{2}}{\log_4 7}}$

$= (\log_2 9)^{2\log^2_{\log_2}9} \times 7^{\frac{1}{2}\log_7 4}$

$= \ 4 \times 2 = 8$

8. The number of 5 digit numbers which are divisible by 4, with digits from the set $\{1,2,3,4,5\}$ and the repetition of digits is allowed, is _____.

Solution: (625)

Option for last two digits are $(12), (24), (32), (44)$ are (52).

$\therefore$ Total number of digits

$= \ 5 \times 5 \times 5 \times 5 = 625$

9. Let X be the set consisting of the first 2018 terms of the arithmetic progression $1, 6, 11, \ldots$, and be the set consisting of the first 2018 terms of the arithmetic progression $9, 16, 23, \ldots$. Then, the number of elements in the set $X \cup Y$ is___.

Solution: (3748)

$X: 1,\ 6,\ 11,\ \ldots,\ 10086$

$Y: 9,\ 16,\ 23, \ldots,\ 14128$

$X \cap Y: 16,\ 51,\ 86, \ldots$

Let $m = n(X \cap Y)$

$\therefore \ 16 + (m-1) \times 35 \leq 10086$

$\Rightarrow \ m \leq 288.71$

$\Rightarrow \quad m = 288$

$\therefore \quad n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$

$= \quad 2018 + 2018 - 288 = 3748$

10. The number of real solutions of the equation $\sin^{-1}\left(\sum_{i=1}^{\infty} x^{i+1} - x\sum_{i=1}^{\infty}\left(\frac{x}{2}\right)^i\right) = \frac{\pi}{2} - \cos^{-1}\left(\sum_{i=1}^{\infty}\left(-\frac{x}{2}\right)^i - \sum_{i=1}^{\infty}(-x)^i\right)$ lying in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$ is____.

(Here, the inverse trigonometric functions $\sin^{-1}x\ and \cos^{-1}x$ assume values in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $[0, \pi]$ respectively.)

Solution: (2)

$$\sum_{i=1}^{\infty} x^{i+1} = \frac{x^2}{1-x}$$

$$\sum_{i=1}^{\infty}\left(\frac{x}{2}\right)^i = \frac{x}{2-x}$$

$$\sum_{i=1}^{\infty}\left(-\frac{x}{2}\right)^i = \frac{-x}{2+x}$$

$$\sum_{i=1}^{\infty}(-x)^i = \frac{-x}{1+x}$$

To have real solutions

$$\sum_{i=1}^{\infty} x^{i+1} - x\sum_{i=1}^{\infty}\left(\frac{x}{2}\right)^i = \sum_{i=1}^{\infty}\left(-\frac{x}{2}\right)^i - \sum_{i=1}^{\infty}(-x)^i$$

$$\frac{x^2}{1-x} - \frac{x^2}{2-x} = \frac{-x}{2+x} + \frac{x}{1+x}$$

$$x\left(x^3 + 2x^2 + 5x - 2\right) = 0$$

$\therefore \quad x = 0$ and let $f(x) = x^3 + 2x^2 + 5x - 2$

$$f\left(\frac{1}{2}\right).f\left(-\frac{1}{2}\right) < 0$$

Hence two solutions exist

11. For each positive integer n, let $y_n = \frac{1}{n}\left((n+1)(n+2)\ldots(n+n)^{\frac{1}{n}}\right)$ for $x \in \mathbb{R}$, let$[x]$ be the greater less than or equal to x. If $\lim\limits_{n\to\infty} y_n = L$, then the value of $[L]$ is_____.

Solution: (1)

$$y_n = \left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\left(1+\frac{n}{n}\right)\right\}^{\frac{1}{n}}$$

$$y_n = \prod_{r=1}^{n}\left(1+\frac{r}{n}\right)^{\frac{1}{n}}$$

$$\log y_n = \frac{1}{n}\sum_{r=1}^{n}\ln\left(1+\frac{r}{n}\right)$$

$$\Rightarrow \lim_{n\to\infty}\log y_n = \lim_{x\to\infty}\sum_{r=1}^{n}\frac{1}{n}\ln\left(1+\frac{r}{n}\right)$$

$$\Rightarrow \log L = \int_0^1 \ln(1+x)dx$$

$$\Rightarrow \log L = \log\frac{4}{e}$$

$$\Rightarrow L = \frac{4}{e}$$

$$[L] = 1$$

12. Let $\vec{a}$ *and* $\vec{b}$ be two unit vectors such that $\vec{a}\,.\,\vec{b} = 0$. For some $x, y \in \mathbb{R}$ let $\vec{c} = x\vec{a} + y\vec{b} + \left(\vec{a}\times\vec{b}\right)$. If $\left|\vec{c}\right| = 2\Big|$ *and* the vector $\vec{c}$ is inclined at the same angle α to both $\vec{a}$ and $\vec{b}$ then the value of $8\cos^2\alpha$ is ____.

Solution: (3)

$\vec{c} = x\vec{a} + y\vec{b} + \vec{a} \times \vec{b}$

$\vec{c}\,.\,\vec{a} = x \; and \; x = 2\cos\alpha$

$\vec{c}\,.\,\vec{b} = y \; and \; y = 2\cos\alpha$

Also, $\left|\vec{a} \times \vec{b}\right| = 1$

$\therefore \quad \vec{c} = 2\cos\left(\vec{a} + \vec{b}\right) + \vec{a} \times \vec{b}$

$\vec{c}^2 = 4\cos^2\alpha\left(\vec{a} + \vec{b}\right)^2 + \left(\vec{a} \times \vec{b}\right)^2 + 2\cos\alpha\left(\vec{a} + \vec{b}\right).\left(\vec{a} \times \vec{b}\right)$

$4 = 8\cos^2\alpha + 1$

$8\cos^2\alpha = 3$

13. Let a, b, c be three non-zero real numbers such that the equation $\sqrt{3} = a\cos x + 2\,b\sin x = c,\ x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, has two distinct real roots α and β with $\alpha + \beta = \frac{\pi}{3}$. Then, the value of $\frac{b}{a}$ is_____.

Solution: (0.5)

$\sqrt{3}\cos x + \frac{2b}{a}\sin x = \frac{c}{a}$

Now, $\sqrt{3}\cos\alpha + \frac{2b}{a}\sin\alpha = \frac{c}{a}$ ….. (i)

$\sqrt{3}\cos\beta + \frac{2b}{a}\sin\beta = \frac{c}{a}$ ….. (ii)

$\sqrt{3}\left[\cos\alpha - \cos\beta\right] + \frac{2b}{a}\left(\sin\alpha - \sin\beta\right) = 0$

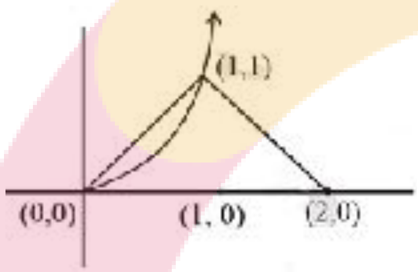
$\sqrt{3}\left[-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right] + \frac{2b}{a}\left[2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) = 0\right]$

$-\sqrt{3} + 2\sqrt{3}\,.\,\frac{b}{a} = 0$

$$\frac{b}{a} = \frac{1}{2} = 0.5$$

14. A farmer F_1 has a land in the shape of a triangle with vertices at $P(0,0),\ Q(1,1)\ and\ R(2,0)$. From this land, a neighbouring farmer F_2 takes away the region which lies between the side PQ and a curve of the form $y = x^n\ (n > 1)$. If the area of the region taken away by the farmer F_2 is exactly 30% of the area of ΔPQR, then the value of n is _____

Solution: (4)



Area $= \int_0^1 (x - x^n)dx = \frac{3}{10}$

$$\left[\frac{x^2}{2} - \frac{x^{n+1}}{n+1}\right]_0^1 = \frac{3}{10}$$

$$\frac{1}{2} - \frac{1}{n+1} = \frac{3}{10} \qquad \therefore\ n + 1 = 5$$

$$\Rightarrow\ n = 4$$

Paragraph "X"

15. Let S be the circle in the xy - plane defined by the equation $x^2 + y^2 = 4$.

Let E_1E_2 and F_1F_2 be the chords of S passing through the point $P_0\ (1,1)$ and parallel to the x - axis and the y - axis , respectively. Let G_1G_2 be the chord of S passing through P_0 and having slope −1. Let the tangents to $S\ at\ E_1$ and E_2 meet at E_3, the tangents to S at F_1 and F_2 meet at F_3, and the tangents to S at G_1 and G_2 meet at G_3. Then, the points $E_3,\ F_3,$ and G_3 lie on the curve

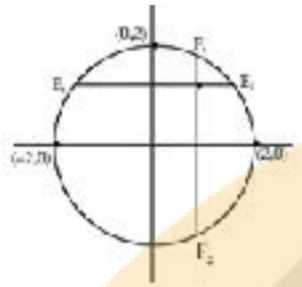
(A) $x + y = 4$

(B) $(x - 4)^2 + (y - 4)^2 = 16$

(C) $(x-4)(y-4)=4$

(D) $xy=4$

Solution: (A)



Co - ordinates of E_1 and E_2 are obtained by solving $y=1$ and $x^2+y^2=4$

$\therefore \quad E_1\left(-\sqrt{3},\ 1\right)$ and $E_2\left(\sqrt{3},\ 1\right)$

Co - ordinates of F_1 and F_2 are obtained by solving

$x=1$ and $x^2+y^2=4$

$F_1\left(1,\sqrt{3}\right)$ and $F_2\left(1,-\sqrt{3}\right)$

Tangent at E_1: $-\sqrt{3}x+y=4$

Tangent at E_2: $-\sqrt{3}x+y=4$

$\therefore E_3(0,4)$

Tangent at F_1: $x+\sqrt{3}y=4$

Tangent at F_2: $x-\sqrt{3}y=4$

$\therefore \quad F_3(4,0)$

And similarly $G_3(2,2)$

$(0,4),(4,0)$ and $(2,2)$ lies on $x+y=4$

Paragraph "X"

16. Let S be the circle in the xy - plane defined by the equation $x^2+y^2=4$.

Let P be a point on the circle S with both coordinates being positive. Let the tangent to S at P intersect the coordinate axes at the points M and N. Then, the mid-point of the line segment MN must lie on the curve

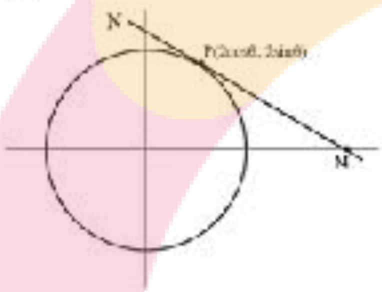
(A) $\left(x+y\right)^2 = 3xy$

(B) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2^{\frac{4}{3}}$

(C) $x^2 + y^2 = 2xy$

(D) $x^2 + y^2 = x^2y^2$

Solution: (D)



Tangent at $P\left(2\cos\theta, 2\sin\theta\right) is\ x\cos\theta + y\sin\theta = 2$

$M(2\sec\theta, 0)$ and $N(0,2\text{cosce}\theta)$

Let midpoint be (h, k)

$h = \sec\theta, k = \text{cosce}\theta$

$$\frac{1}{h^2} + \frac{1}{k^2} = 1$$

$$\frac{1}{x^2} + \frac{1}{y^2} = 1$$

Paragraph "A"

17. There are five students $S_1,\ S_2,\ S_3,\ S_4$ and S_5 in a music class and for them there are five seats $R_1,\ R_2,\ R_3,\ R_4$ and R_5 arranged in a row, where initially the seat R_i is allotted to the student $S_i,\ i\ =\ 1,2,3,4,5$. But, on the examination day, the five students are randomly allotted the five seats

The probability that, on the examination day, the student S_1 gets the previously allotted seat R_1, and NONE of the remaining students gets the seat previously allotted to him/her is

(A) $\frac{3}{40}$

(B) $\frac{1}{8}$

(C) $\frac{7}{40}$

(D) $\frac{1}{5}$

Solution: (A)

Required probability $= \frac{4!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)}{5!} = \frac{9}{120} = \frac{3}{40}$

Paragraph "A"

18. There are five students $S_1,\ S_2,\ S_3,\ S_4$ and S_5 in a music class and for them there are five seats $R_1,\ R_2,\ R_3,\ R_4$ and R_5 arranged in a row, where initially the seat R_i is allotted to the student $S_i,\ i = 1,2,3,4,5$. But, on the examination day, the five students are randomly allotted the five seats

For $i = 1,2,3,4,$ let T_i denote the event that the students $S_i\ and\ S_{i+1}$ do NOT sit adjacent to each other on the day of the examination. Then, the probability of the event $T_1 \cap\ T_2 \cap\ T_3 \cap T_4$ is

(A) $\frac{1}{15}$

(B) $\frac{1}{10}$

(C) $\frac{7}{60}$

(D) $\frac{1}{5}$

Solution: (C)

$$n\left(T_1 \cap T_2 \cap T_3 \cap T_4\right) = Total - n\left(\overline{T_1} \cup \overline{T_2} \cup \overline{T_3} \cup \overline{T_4}\right)$$

$$= 5! - \left({}^4C_1 4!2! - \left({}^3C_1 . 3!2! + {}^3C_1 3!2!2!\right) + \left({}^2C_1 2!2! + {}^4C_1 . 2.2!\right) - 2\right)$$

$= \quad 14$

Probability $= \dfrac{14}{5!} = \dfrac{7}{60}$