Permutation & Combination Formulas

**Permutation Formulas**

★ When repetition is not allowed: P is a permutation or arrangement of r things from a set of n things without replacement. We define P as:

\[ nP_r = \frac{n!}{(n-r)!} \]

★ When repetition is allowed: P is a permutation or arrangement of r things from a set of n things when repetition is allowed. We define P as:

\[ nP_r = n^r \]

**Derivation of Permutation Formula:**

Let us assume that there are r boxes and each of them can hold one thing. There will be as many permutations as there are ways of filling in r vacant boxes by n objects.

- No. of ways the first box can be filled: \( n \)
- No. of ways the second box can be filled: \( (n - 1) \)
- No. of ways the third box can be filled: \( (n - 2) \)
- No. of ways the fourth box can be filled: \( (n - 3) \)
- No. of ways \( r^{th} \) box can be filled: \( [n - (r - 1)] \)

The number of permutations of \( n \) different objects taken \( r \) at a time, where \( 0 < r \leq n \) and the objects do not repeat is: \( n(n - 1)(n - 2)(n - 3) \ldots (n - r + 1) \)

\[ \Rightarrow nP_r = n(n - 1)(n - 2)(n - 3) \ldots (n - r + 1) \]

Multiplying and dividing by \( (n - r) (n - r - 1) \ldots 3 \times 2 \times 1 \), we get:

\[ nP_r = \frac{[n(n-1)(n-2)(n-3)\ldots(n-r+1)(n-r)(n-r-1)\ldots3\times2\times1]}{(n-r)(n-r-1)\ldots3\times2\times1} = \frac{n!}{(n-r)!} \]
\[ nP_r = \frac{n!}{(n-r)!} \]

**Combination Formulas**

★ When repetition is not allowed: C is a combination of n distinct things taking r at a time (order is not important). We define C as:

\[ nC_r = \frac{nP_r}{r!} = \frac{n!}{(n-r)!r!} \]

★ When repetition is allowed: C is a combination of n distinct things taking r at a time (order is not important) with repetition. We define C as:

\[ nCr = \frac{(n + r - 1)!}{[r!(n - 1)!]} \]

**Derivation of Combination Formula:**

Let us assume that there are r boxes and each of them can hold one thing.

– No. of ways to select the first object from n distinct objects: \( n \)

– No. of ways to select the second object from \((n-1)\) distinct objects: \( (n-1) \)

– No. of ways to select the third object from \((n-2)\) distinct objects: \( (n-2) \)

– No. of ways to select \(r^{th}\) object from \([n-(r-1)]\) distinct objects: \([n-(r-1)]\)

Completing the selection of r things from the original set of n things creates an ordered subset of r elements.

\[ \therefore \text{The number of ways to make a selection of } r \text{ elements of the original set of } n \text{ elements is: } n \]
\[ (n - 1) \ (n - 2) \ (n-3) \ldots (n - (r - 1)) \text{ or } n \ (n - 1) \ (n - 2) \ldots (n - r + 1). \]

Let us consider the ordered subset of r elements and all its permutations. The total number of all permutations of this subset is equal to \( r! \) because \( r \) objects in every combination can be rearranged in \( r! \) ways.

Hence, the total number of permutations of n different things taken r at a time is \( nC_r \times r! \). It is nothing but \( nP_r \).

\[ nP_r = nC_r \times r! \]
\[ nC_r = \frac{nP_r}{r!} = \frac{n!}{(n-r)!r!} \]