1. Sum of an infinite G. P. with first term $a$ and common ratio $r$ is 15 and the sum of the squares of the terms of the same G. P. is 150. The sum of an infinite G. P. with first term $a$ and common ratio $r^2$ is:

(A) $\frac{25}{2}$

(B) $\frac{1}{5}$

(C) $\frac{27}{5}$

(D) $\frac{23}{5}$

Answer: (A)

Solution: Give, that

\[ a + ar + ar^2 \ldots = 15 \]

\[ \frac{a}{1-r} = 15 \ldots (i) \]

\[ a^2 + a^2r^2 + a^2r^4 \ldots = 150 \]

\[ \frac{a^2}{1-r^2} = 150 \ldots (ii) \]

From equation (i) and (ii)

\[ \frac{(1-r)^2}{1-r^2} = \frac{150}{225} \]

\[ \Rightarrow 3 + 3r^2 - 6r = 2 - 2r^2 \]

\[ \Rightarrow 5r^2 - 6r + 1 = 0 \]

\[ \Rightarrow r = \frac{1}{5}, \frac{1}{5} \]

Now, $r = \frac{1}{5}$ and $a = 12, r \neq 1$ as it is infinite G. P.

\[ a + ar^2 + ar^4 \ldots = \frac{a}{1-r^2} \]

\[ = \frac{12}{1 - \frac{1}{25}} = \frac{25}{2} \]

2. If $(1 + x)^{20} = \sum_{r=0}^{20} \binom{20}{r}x^r$, then the value of $\sum_{r=0}^{20} r^2 \binom{20}{r}$ is:

(A) $2^{20} (105)$

(B) $2^{18} (105)$

(C) $2^{19} (205)$

(D) $2^{20} (95)$

Answer: (A)
Solution:

\[(1 + x)^{20} = \sum_{r=0}^{20} C_r x^r\]

Now \(\sum_{r=0}^{20} r^2 C_r\)

\[= \sum_{r=0}^{20} r(r - 1) C_r + \sum_{r=0}^{20} r C_r\]

\[= 20 \times 19 \times \sum_{r=0}^{20} C_{r-2} + 20 \times \sum_{r=0}^{20} C_{r-1} \quad \text{(using} \frac{n}{r} C_r = \frac{n-1}{r-1} C_{r-1}\text{)}\]

\[= 380 \times 2^{18} + 20 \times 2^{19}\]

\[= 420 \times 2^{18}\]

\[= (105)2^{20}\]

3. If probability of independent events \(A\) and \(B\) are \(P(A) = p, P(B) = 2p\) and probability of exactly one of \(A\) and \(B\) is \(\frac{5}{9}\), then the value of \(p\) is

(A) \(\frac{2}{5}\)

(B) \(\frac{1}{3}\)

(C) \(\frac{7}{12}\)

(D) \(\frac{1}{12}\)

Answer: (B)

Solution:

\(P(A) = p, \quad P(B) = 2p\)

\(P\) ( exactly one)= \(\frac{5}{9}\)

\(\Rightarrow P(A) + P(B) - 2P(A \cap B) = \frac{5}{9}\)

\(\Rightarrow p + 2p - 2 \cdot p \cdot 2p = \frac{5}{9} \quad \text{[A and B are independent]}\)

\(\Rightarrow 27p - 36p^2 = 5\)

\(\Rightarrow 36p^2 - 27p + 5 = 0 \Rightarrow p = \frac{5 \pm \frac{1}{3}}{12}\)

4. The value of integration \(\int_{-1/2}^{1/2} \left[ \left(\frac{x-1}{x+1}\right)^2 + \left(\frac{x+1}{x-1}\right)^2 - 2 \right]^{1/2} dx\) is:
(A) \(4\ell n\left(\frac{3}{2}\right)\)

(B) \(4\ell n\left(\frac{3}{4}\right)\)

(C) \(4\ell n\left(\frac{4}{3}\right)\)

(D) \(2\ell n\left(\frac{3}{2}\right)\)

Answer: (C)

Solution:

\[
\int_{-1/2}^{1/2} \left[ \frac{(x-1)^2 + (x+1)^2 - 2}{x^2 - 1} \right]^{1/2} dx
\]

\[
\Rightarrow \int_{-1/2}^{1/2} \frac{x - 1 - x + 1}{x + 1} \frac{x + 1 - x - 1}{x - 1} dx
\]

\[
= \int_{-1/2}^{1/2} \frac{(x - 1)^2 - (x + 1)^2}{x^2 - 1} dx
\]

\[
= \int_{-1/2}^{1/2} \frac{-4x}{x^2 - 1} dx
\]

\[
= 8 \int_0^{1/2} \frac{|x|}{|x^2 - 1|} dx \quad \text{(as } \frac{|x|}{|x^2 - 1|} \text{ is an even function)}
\]

\[
= 4 \int_0^{1/2} \frac{2x}{1-x^2} dx
\]

\[
= 4 \int_1^{3/4} -\frac{dt}{t} \quad 1 - x^2 = t \Rightarrow -2dx = dt
\]

\[
= 4[\ln(t)]_{1/3}^{3/4}
\]

\[
= 4 \left[ \ln(1) - \ln\left(\frac{3}{4}\right) \right]
\]

\[
= 4\ln\left(\frac{4}{3}\right)
\]

5. Let arg \(\frac{z+1}{z-1}\) = \(\frac{\pi}{4}\). Then the locus of \(z\) is a circle whose radius and centre respectively are:

(A) \(\sqrt{2}, \ (0,1)\)

(B) \(\sqrt{2}, \ (0,-1)\)

(C) \(\sqrt{2}, \ (0,0)\)

(D) 1, \ (1,1)

Answer: (B)

Solution:
\[ \arg \left( \frac{z+1}{z-1} \right) = \frac{\pi}{4} \]

Let \( z = x + iy \)

\[ \Rightarrow \arg \left( \frac{x + iy + 1}{x + iy - 1} \right) = \frac{\pi}{4} \]

\[ \Rightarrow \tan^{-1}\left( \frac{y}{x+1} \right) - \tan^{-1}\left( \frac{y}{x-1} \right) = \frac{\pi}{4} \]

taking tan on both sides, we get

\[ \frac{y}{x+1} - \frac{y}{x-1} = 1 \]

\[ \Rightarrow \frac{-2y}{x^2 - 1 + y^2} = 1 \]

\[ \Rightarrow x^2 - 1 + y^2 = -2y \]

\[ = x^2 + y^2 + 2y - 1 = 0 \]

\[ C(0, -1), r = \sqrt{0 + 1 + 1} = \sqrt{2} \]

6. If \( A \) and \( B \) are two square matrices of order \( 2 \times 2 \) such that \( A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \); \( B = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \) (where \( i = \sqrt{-1} \)) and \( A^T B^{2021} A = Q \), then the value of \( AQA^T \) is

(A) \( \begin{bmatrix} 1 & 0 \\ i & 1 \end{bmatrix} \)

(B) \( \begin{bmatrix} 1 & 0 \\ 2021i & 1 \end{bmatrix} \)

(C) \( \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \)

(D) \( \begin{bmatrix} 1 & 0 \\ 2020i & 1 \end{bmatrix} \)

Answer: (B)

Solution:

Given,

\[ A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \ B = \begin{bmatrix} 1 & 0 \\ i & 1 \end{bmatrix} \]

\[ A^T B^{2021} A = Q \]

\[ B = \begin{bmatrix} 1 & 0 \\ i & 1 \end{bmatrix} \]

\[ B^2 = \begin{bmatrix} 1 & 0 \\ i & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ i & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2i & 1 \end{bmatrix} \]

\[ B^3 = \begin{bmatrix} 1 & 0 \\ 2i & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ i & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3i & 1 \end{bmatrix} \]

\[ B^4 = \begin{bmatrix} 1 & 0 \\ 3i & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ i & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4i & 1 \end{bmatrix} \]
\[ B^{2021} = \begin{bmatrix} 1 & 0 \\ 2021i & 1 \end{bmatrix} \]

Now, given that
\[ A^T B^{2021} A = Q \]
\[ AQA^T = AA^T B^{2021} AA^T \ldots (1) \]

We know \( A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \) \( \Rightarrow A^T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \)
\[ \Rightarrow AA^T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \]

So, Equation (1) will be
\[ AQA^T = IB^{2021}I = B^{2021} \]
\[ = \begin{bmatrix} 1 & 0 \\ 2021i & 1 \end{bmatrix} \]

7. The mean of 20 observations is 10 and standard deviation is 2.5. If one of the observation 25 is replaced by 35 then new mean and standard deviation are \( \alpha \) and \( \sqrt{\beta} \) respectively, then ordered pair \( (\alpha, \beta) \) is

(A) (10.5, 36)
(B) (10.5, 26)
(C) (10.5, 25)
(D) (10.5, 23)

Answer: (B)

Solution:

Given

Mean \( (x_{old}) = 10 \)

S. D. = 2.5

Number of observations \( (n) = 20 \)

\[ \bar{x}_{old} = \frac{x_1 + x_2 + x_3 \ldots x_{19} + 25}{20} = 10 \]
\[ \Rightarrow x_1 + x_2 + x_3 + \ldots x_{19} + 25 = 200 \]
\[ \Rightarrow x_1 + x_2 + x_3 + \ldots x_{19} = 175 \]

As 25 is replaced by 35

\[ \bar{x}_{new} = \frac{x_1 + x_2 + \ldots x_{19} + 135}{20} = \frac{175 + 135}{20} \]
\[
\frac{210}{20} = 10.5
\]

\[(S.D.)_{old} = \sqrt{\frac{x_1^2 + x_2^2 + x_3^2 + \ldots + x_{19}^2 + (25)^2}{20}} - (10)^2
\]

\[\Rightarrow 2.5 = \sqrt{\frac{x_1^2 + x_2^2 + x_3^2 + \ldots + x_{19}^2 + 625}{20}} - 100
\]

\[\Rightarrow x_1^2 + x_2^2 + x_3^2 + \ldots + x_{19}^2 = 1500
\]

\[(S.D.)_{new} = \sqrt{\frac{x_1^2 + x_2^2 + x_3^2 + \ldots + x_{19}^2 + (35)^2}{20}} - (10.5)^2
\]

\[= \sqrt{\frac{1500 + 1225}{20}} - 110.25
\]

\[= \sqrt{26}
\]

\[\Rightarrow (\alpha, \beta) = (10.5, \sqrt{26})
\]

8. If \(\vec{a} = \hat{i} + \hat{j} + \hat{k}, \vec{b} = \hat{i} - \hat{k}, \vec{a} \times \vec{c} = \vec{b}, \vec{a} \cdot \vec{c} = 3\), then the value of \([\vec{a} \vec{b} \vec{c}]\) is

(A) \(\sqrt{2}\)

(B) \(-2\)

(C) \(2\)

(D) \(-\sqrt{2}\)

Answer: (B)

Solution:

\(\vec{a} = \hat{i} + \hat{j} + \hat{k}\)

\(\vec{b} = \hat{i} - \hat{k}\)

\(\vec{a} \times \vec{c} = \vec{b}, \vec{a} \cdot \vec{c} = 3\)

Now, we know

\([\vec{a} \vec{b} \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c})\)

Using properties of scalar Triple product

We can write \([\vec{a} \vec{b} \vec{c}] = [\vec{b} \vec{c} \vec{a}] = \vec{b} \cdot (\vec{c} \times \vec{a})\)

\[\Rightarrow [\vec{a} \vec{b} \vec{c}] = -\vec{b} \cdot (\vec{a} \times \vec{c}) \{\because \vec{c} \times \vec{a} = -\vec{a} \times \vec{c}\}\]

\[= -\vec{b} \cdot \vec{b}\]
9. In a city, 89% people are suffered from diabetes and 98% people are suffered from heart diseases. If \( x \)\% people are suffered from diabetes as well as heart diseases then the possible values of \( x \) cannot be lie in which of the following set:

(A) \{81, 83, 85, 86\}
(B) \{82, 87, 88, 91\}
(C) \{87, 88, 89, 90\}
(D) \{88, 89, 92, 95\}

Answer: (A)

Solution:
Let 
\( D = \) Number of people suffered from diabetes.
\( H = \) Number of people suffered from heart diseases.

Given
\( n(D) = 89\% \)
\( n(H) = 98\% \)
\( n(D \cap H) = x \% \)

We know
\( n(D \cap H) = n(D) + n(H) - n(D \cup H) \)

For possible range of \( n(D \cap H) \)
\( \max \{0, n(D) + n(H) - n(D \cup H)\} \leq n(D \cap H) \leq \min \{n(D), n(H)\} \)
\( \max \{0, (85 + 98 - 100)\%\} \leq x\% \leq \min \{89\%, 98\%\} \)
\( 87\% \leq x \leq 89\% \)

10. If \( \ln(x + y) = 4xy \), then the value of \( \frac{d^2y}{dx^2} \) at \( x = 0 \) is

(A) 30
(B) 40
(C) 35
(D) 38

Answer: (B)

Solution:
Given \( \ln(x + y) = 4xy \ldots (i) \)
Put $x = 0$ in (i) $\ell ny = 0 \Rightarrow y = 1$

from (i)

$\ell n(x + y) = 4xy$

$\Rightarrow x + y = e^{4xy}$

now differentiate wrt $x$

$\Rightarrow 1 + \frac{dy}{dx} = e^{4xy} \left(4y + 4x \frac{dy}{dx}\right) \ldots (ii)$

At $(0,1)$

$\Rightarrow 1 + \frac{dy}{dx} = e^{4xy} \left(4y + 4x \frac{dy}{dx}\right) \ldots (iii)$

$1 + \left(\frac{dy}{dx}\right)_{x=0} = 4 \Rightarrow \left(\frac{dy}{dx}\right) = 3 \ldots (iv)$

Now again differentiate (ii)

$$\frac{d^2y}{dx^2} = e^{4xy} \left(4y + 4x \frac{dy}{dx}\right)^2 + e^{4xy} \left(4\frac{dy}{dx} + 4\frac{dy}{dx} + 4xd^2y\right)$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)_{x=0} = e^0 \left(4 \times 1 + 4 \times 0 \times 3\right)^2 + e^0 \left(4 \times 3 + 4 \times 3 + 4 \times 0 \times \frac{d^2y}{dx^2}\right)$$

$$= (4)^2 + 24 = 16 + 24 = 40$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=0} = 40$$

11. If locus of a variable point $P$, whose sum of squares of distances from points $(0,0), (0,1), (1,0)$ and $(1,1)$ is 18, is a circle of diameter $d$, then value of $d^2$ is equal to

Answer: 16.00

Solution:

Let $P(h,k)$

Given,

Sum of square of distances of $P$ from given points = 18

$\Rightarrow (h - 0)^2 + (k - 0)^2 + (h - 0)^2 + (k - 1)^2 + (h - 1)^2 + (k - 0)^2 + (h - 1)^2 + (k - 1)^2 = 18$

$\Rightarrow h^2 + k^2 + h^2 + k^2 - 2k + 1 + h^2 - 2h + 1 + k^2 + h^2 - 2h + 1 + k^2 - 2k + 1 = 18$

$\Rightarrow 4h^2 + 4k^2 - 4h - 4k + 4 = 18$

$\Rightarrow h^2 + k^2 - h - k - \frac{14}{4} = 0$

Replace $h \rightarrow x$ and $k \rightarrow y$

$x^2 + y^2 - x - y - \frac{14}{4} = 0$
Centre = \( \left( \frac{1}{2}, \frac{1}{2} \right) \)

Radius \( r \) = \( \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{14}{4}} = \sqrt{4} = 2 \)

d = 2r = 2 \times 2 = 4

So, \( d^2 = 16 \)

12. If \((1 + y)\tan^2 x + \tan x \frac{dy}{dx} + y = 0\) and \( \lim_{x \to 0^+} xy = 1 \), then the value of \( y \left( \frac{\pi}{4} \right) \) is

(A) \( \frac{\pi}{4} \)

(B) 0

(C) \( -\frac{\pi}{4} \)

(D) \( -\frac{\pi}{2} \)

Answer: (A)

Solution:

\[(1 + y)\tan^2 x + \tan x \frac{dy}{dx} + y = 0\]

Dividing by \( \tan x \)

\[\frac{dy}{dx} + (1 + y)\tan x = -y \cot x\]

\[\frac{dy}{dx} + (\tan x + \cot x)y = -\tan x ...... (i)\]

It's a first order linear differential equation

\[I.F. = e^{\int (\tan x + \cot x)dx}\]

\[= e^{\int \frac{\tan^2 x + 1}{\tan x}}dx\]

\[= e^{\int \frac{\sec^2 x}{\tan x}dx} = e^{\ln \tan x} = \tan x\]

So equation (i) will be come

\[y \tan x = \int -\tan^2 x \, dx + C\]

\[y \tan x = \int (1 - \sec^2 x) \, dx + C\]

\[y \tan x = \int dx - \int \sec^2 x \, dx + C\]

\[y \tan x = x - \tan x + C\]

Given \( \lim_{x \to 0^+} xy = 1 \)

\[\Rightarrow \lim_{x \to 0^+} \left( \frac{x}{\tan x} \right) (x - \tan x + C) = 1\]

\[\Rightarrow 1(0 - 0 + C) = 1\]
⇒ C = 1
So the function is
\[ y \tan x = x - \tan x + 1 \]
\[ y \tan x = x - \tan x + 1 \]
\[ y \left( \frac{\pi}{4} \right) \Rightarrow y \tan \frac{\pi}{4} = \frac{\pi}{4} - \tan \frac{\pi}{4} + 1 \]
\[ y \cdot 1 = \frac{\pi}{4} - 1 + 1 \]
\[ y = \frac{\pi}{4} \]

13. How many 3 digit numbers are possible using the digits 0,1,3,4,6,7 if repetition is allowed.
Answer: 180
Solution: The hundreds place can be filled in 5 ways, tens place can be filled in 6 ways and units place can be filled in 6 ways.
\[ 5 \times 6 \times 6 = 180 \]

14. If \( \frac{\cos x}{1 + \sin x} = |\tan x| \), then number of solutions in \([0, 2\pi]\) is
Answer: (1)
Solution:
\[ \frac{\cos x}{1 + \sin x} = |\tan x| \{ \sin x \neq -1 \} \]
Case-I \( \tan x \geq 0 \)
\[ \Rightarrow \frac{\cos x}{1 + \sin x} = \frac{\sin x}{\cos x} \]
\[ \Rightarrow \cos^2 x = \sin x + \sin^2 x \]
\[ \Rightarrow 2\sin^2 x + \sin x - 1 = 0 \]
\[ \Rightarrow \sin x = \frac{1}{2}, -1 \]
\[ \Rightarrow \sin x = \frac{1}{2} \]
\[ \sin x = -1 \text{ not possible} \]
\[ \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6} \] (rejected)
\[ \Rightarrow 1 \text{ Solution i.e. } \frac{\pi}{6} \]
Case-II \( \tan x < 0 \)
\[ \frac{\cos x}{1 + \sin x} = -\tan x \]
\[ \Rightarrow \cos^2 x = -\sin x - \sin^2 x \]
⇒ \sin x = -1 \text{ (not possible)}

15. A wire of length 26 units is cut into two parts which are bent respectively to form a square and a circle. If the sum of areas of square and the circle so formed is minimum, then the circumference of circle is

(A) \(\frac{36\pi}{\pi+4}\)
(B) \(\frac{18\pi}{\pi+4}\)
(C) \(\frac{9\pi}{\pi+2}\)
(D) \(\frac{3\pi}{\pi+9}\)

Answer: (A)

Solution:

Let the radius of circle is \(r\) and side of square is \(a\).

Given, that

\[2\pi r + 4a = 36\]

⇒ \[r = \frac{18 - 2a}{\pi}\]

Now sum of areas of circle and square

\[A = \pi r^2 + a^2\]

\[A = \frac{(18 - 2a)^2}{\pi} + a^2\]

\[\frac{dA}{da} = \frac{2(18 - 2a)(-2)}{\pi} + 2a\]

\[\frac{dA}{da} = 0 \Rightarrow 36 - 4a = a\pi \Rightarrow a = \frac{36}{\pi + 4}\]

\[\frac{d^2A}{da^2} = \frac{8}{\pi} + 2 > 0\]

So, the area is minimum

Now, \(r = \frac{18 - 2a}{\pi} = \frac{18 - \frac{72}{\pi + 4}}{\pi} = \frac{18}{\pi + 4}\)

So, circumference = \(2\pi r\)

= \(2\pi \times \frac{18}{\pi + 4} = \frac{36\pi}{\pi + 4}\) units

16. Tangent at a point P on the ellipse \(\frac{x^2}{8} + \frac{y^2}{4} = 1\) is perpendicular to \(x + 2y = 0\). Then the value of \((5 - e^2) \times \text{Area of } \triangle SPS'\) is, where e is eccentricity, \(S\) & \(S'\) are foci and P is in second quadrant

Answer: 6

Solution:
\[ \frac{x^2}{8} + \frac{y^2}{4} = 1 \Rightarrow e = \sqrt{1 - \frac{4}{8}} = \frac{1}{\sqrt{2}} \Rightarrow ae = 2 \]

Tangent perpendicular to \( x + 2y = 0 \)

\( y = mx \pm \sqrt{8m^2 + 4} \quad \{m = 2\} \)

\( y = 2x \pm 6 \) \{ P in 2^{nd} quadrant\}

\( y = 2x + 6 \Rightarrow 2x - y = -6 \)

Now comparing with

\[ \frac{xx}{8} + \frac{yy}{4} = 1 \], we get,

\[ \frac{x_1}{16} = \frac{y_1}{-4} = \frac{1}{-6} \]

\( P\left(-\frac{8}{3}, \frac{2}{3}\right) \)

\( (\text{Area of SPS'}) \times (5 - e^2) \)

\[ = \frac{1}{2} \times 4 \times 2 \times \left( 5 - \frac{1}{2} \right) \]

\[ = 6 \text{ Sq. Units.} \]

17. If \( f(x) = \cos \left[ 2\tan^{-1} \left( \sin \left( \cot^{-1} \sqrt{\frac{1-x}{x}} \right) \right) \right] \), then

(A) \( f'(x)(1-x)^2 - 2f^2(x) = 0 \)

(B) \( f'(x)(x-1)^2 + 2f(x) = 0 \)

(C) \( f'(x)(1-x)^2 + 2f^2(x) = 0 \)

(D) \( f'(x)(x+1)^2 - 2f(x) = 0 \)

Answer: (C)

Solution: Let \( \cot^{-1} \sqrt{\frac{1-x}{x}} = \theta \Rightarrow \cot \theta = \sqrt{\frac{1-x}{x}} \Rightarrow \sin \theta = \sqrt{x} \)

Now, \( f(x) = \cos \left[ 2\tan^{-1} \left( \sin \theta \right) \right] \)

\( f(x) = \cos \left[ 2\tan^{-1} \sqrt{x} \right] \)

Let \( \tan^{-1}\sqrt{x} = \alpha \)
\[ \sqrt{x} = \tan \alpha \]
So, \( f(x) = \cos (2\alpha) \)

\[
f(x) = \frac{1-x}{1+x}
\]

\[
f'(x) = \frac{(1+x)(-1)-(1-x)(1)}{(1+x)^2}
\]

\[
f''(x) = \frac{-2}{(1+x)^2}
\]

Multiplying by \((1 - x)^2\), we get

\[
f'(x)(1 - x)^2 = \frac{-2}{(1 + x)^2} (1 - x)^2
\]

18. Find the value of \( \frac{1}{1+x} + \frac{2}{1+x^2} + \frac{2^2}{1+x^4} + \cdots + \frac{2^{100}}{1+x^{200}} \) at \( x = 2 \)

(A) \( \frac{2^{200}}{1+2^{400}} - 1 \)

(B) \( \frac{2^{101}}{1-2^{400}} + 1 \)

(C) \( \frac{2^{100}}{1-2^{400}} - 1 \)

(D) \( \frac{2^{100}}{1+2^{200}} - 1 \)

Answer (B)

Solution: Given \( \frac{1}{1+x} + \frac{2}{1+x^2} + \frac{2^2}{1+x^4} + \cdots + \frac{2^{100}}{1+x^{200}} \)

add and subtract \( \frac{1}{1-x} \), we get

\[
\frac{1}{1-x} + \frac{1}{1+x} + \frac{2}{1+x^2} + \frac{2^2}{1+x^4} + \cdots + \frac{2^{100}}{1+x^{200}} - \frac{1}{1-x}
\]

\[
= \frac{2}{1-x^2} + \frac{2}{1+x^2} + \frac{2^2}{1+x^4} + \cdots + \frac{2^{10}}{1+x^{200}} - \frac{1}{1-x}
\]

\[
= \frac{2^2}{1-x^4} + \frac{2^2}{1+x^4} + \cdots + \frac{2^{100}}{1+x^{200}} - \frac{1}{1-x}
\]

similarly we will get

\[
\frac{2^{101}}{1-x^{400}} - \frac{1}{1-x}
\]

At \( x = 2 \), we get,

\[
\frac{2^{101}}{1 - 2^{400}} + 1
\]
19. A circle with centre \((-15,0)\) and radius equal to \(\frac{15}{2}\). Chord to circle through \((-30,0)\) is tangent to \(y^2 = 30x\). Find the length of the chord.

(A) \(\frac{15}{\sqrt{5}}\)

(B) \(\frac{15}{4}\)

(C) \(\frac{15}{\sqrt{3}}\)

(D) \(\frac{15}{\sqrt{7}}\)

Answer (A)

Solution: Given

\[ y^2 = 30x \]

\[ 4a = 30 \Rightarrow a = \frac{15}{2} \]

Equation of tangent

\[ y = mx + \frac{a}{m} \]

\[ y = mx + \frac{15}{2m} \]

Passes through \((-30,0)\)

\[ 0 = -30m + \frac{15}{2m} \]

\[ 30m = \frac{15}{2m} \Rightarrow m^2 = \frac{1}{4} \Rightarrow m = \pm \frac{1}{2} \]

Case I: \(m = \frac{1}{2}\)

\[ y = \frac{x}{2} + 15 \]

Distance of line from centre is

\[ d_1 = \frac{\left|\frac{15}{2} + 15\right|}{\sqrt{1^2 + (\frac{1}{2})^2}} = \frac{15}{\sqrt{5}} = \frac{15}{\sqrt{5}} \]

Case II: \(m = -\frac{1}{2}\)

\[ y = -\frac{x}{2} - 15 \]

Distance of line from centre is

\[ d_2 = \frac{\left|\frac{15}{2} - 15\right|}{\sqrt{1^2 + (\frac{1}{2})^2}} = \frac{15}{\sqrt{5}} \]

\[ \Rightarrow d_1 = d_2 = d = \frac{15}{\sqrt{5}} \]

Length of chord = \(2\sqrt{r^2 - d^2}\)
\[
= 2\sqrt{\left(\frac{15}{2}\right)^2 - \left(\frac{15}{\sqrt{5}}\right)^2}
\]
\[
= 2 \times 15 \sqrt{\frac{1}{4} - \frac{1}{5}}
\]
\[
= 30 \times \sqrt{\frac{1}{5}}
\]
\[
= \frac{30 \sqrt{5}}{\sqrt{20}} = \frac{15 \sqrt{5}}{\sqrt{5}}
\]

20. The value of \( \lim_{n \to \infty} \frac{1}{n} \sum_{r=0}^{2n-1} \frac{n^2 + 4r^2}{n} \) is

(A) \( \tan^{-1} 4 \)

(B) \( \tan^{-1} 2 \)

(C) \( \frac{1}{2} \tan^{-1} 4 \)

(D) \( \frac{1}{2} \tan^{-1} 2 \)

Answer: (C)

Solution:

\[
\frac{1}{n} \sum_{r=0}^{2n-1} \frac{1}{1 + 4r^2} \cdot \frac{1}{n}
\]
\[
= \int_0^2 \frac{1}{1 + 4x^2} \, dx \quad \text{let} \quad \frac{r}{n} = x, \quad \frac{1}{n} = dx
\]
\[
= \frac{1}{2} \left( \tan^{-1}(2x) \right)_0^2
\]
\[
= \frac{1}{2} \left( \tan^{-1} 4 - \tan^{-1} 0 \right)
\]
\[
= \frac{1}{2} \tan^{-1} 4
\]

21. If \( z = \frac{-1+\sqrt{3}i}{2} \), then the value of \( 21 + \left( z + \frac{1}{z} \right)^3 + \left( z^2 + \frac{1}{z^2} \right)^3 + \left( z^3 + \frac{1}{z^3} \right)^3 \ldots + \left( z^{21} + \frac{1}{z^{21}} \right)^3 \) is equal to

Ans: (63)

Solution: Here, \( z = \omega \), so \( 1 + \omega + \omega^2 = 0 \) and \( \omega^3 = 1 \).

Now, \( z + \frac{1}{z} = \omega + \frac{1}{\omega} = \omega + \omega^2 = -1 \)

Similarly, \( z^2 + \frac{1}{z^2} = \omega^2 + \frac{1}{\omega^2} = \omega + \omega^2 = -1, z^3 + \frac{1}{z^3} = \omega^3 + \frac{1}{\omega^3} = 2 \)

Hence, \( 21 + \left( z + \frac{1}{z} \right)^3 + \left( z^2 + \frac{1}{z^2} \right)^3 + \left( z^3 + \frac{1}{z^3} \right)^3 \ldots + \left( z^{21} + \frac{1}{z^{21}} \right)^3 = 21 + (-1 - 1 + 8) + 6 (6 \text{ times}) = 63 \)