JEE Main 2021 August 31, Shift 2 (Mathematics)

1. Find the number of 4 digit numbers which are neither divisible by 7 nor 3 .

Ans. 5143

Sol. Total 4 digit number = $\underline{9} \times \underline{10} \times \underline{10} \times \underline{10} = 9000$

4 digit numbers divisible by $7(n_7)$

$1001, 1008, \ldots \ldots \ldots \ldots . .9996$

Applying formula of n^{th} term of an AP.

$T_n = a + (n-1)d$

$9996 = 1001 + (n_7 - 1)7$

$n_7 = 1286$

4 digit number divisible by $3(\mathrm{n}_3)$

$1002, 1005, \ldots \ldots . , 9999$

$9999 = 1002 + (n_3 - 1)3$

Applying formula of n^{th} term of an AP.

$T_n = a + (n-1)d$

$\mathrm{n}_3 = 3000$

4 digit numbers divisible by $21(\mathrm{n}_{21})$

$1008, 1031, \ldots \ldots \ldots , 9996$

Applying formula of n^{th} term of an AP.

$T_n = a + (n-1)d$

$9996 = 1008 + (n_{21} - 1)21$

$\mathrm{n}_{21} = 429$

so, 4 digit numbers neither divisible by 7 nor 3

$= 9000 - 1286 - 3000 + 429$

$= 5143$

2. If $\lim_{x\to 0}(\cos x)^{\cot x} = A$ and $\lim_{x\to\pi/4}\frac{\tan^3 x-\tan x}{\cos\left(x+\frac{\pi}{4}\right)} = B$. Given that A & B are the roots of equation $ax^2 + bx - 4 = 0$. Then the value of a and b respectively are

(A) $1, 3$

(B) $1,\ -4$

(C) $1,\ -3$

(D) $-1,\ 3$

Ans. (A)

Sol. Given

$\lim_{x\to 0}(\cos\ x)^{\cot x} = A$ and $\lim_{x\to\pi/4}\frac{\tan^3 x-\tan x}{\cos\left(x+\frac{\pi}{4}\right)} = B$

Now,

$A = \lim_{x\to 0}(\cos x)^{\cot x}$

$= \lim_{x\to 0}(\cos x)^{\frac{1}{\tan x}}$

$= e^{\lim_{x\to 0}\frac{\cos x-1}{\tan x}}$

Applying limits using L'hospital rule

$= e^{\lim_{x\to 0}\frac{-\sin x}{\sec^2 x}}$

$A = e^0 = 1$

Now,

$B = \lim_{x\to\pi/4}\frac{\tan^3 x-\tan x}{\cos\left(x+\frac{\pi}{4}\right)}$

Applying limits using L'hospital rule, we get

$$= 2\lim_{x\to\pi/4}\frac{(3\tan^2 x - 1)\sec^2 x}{-\sin\left(x+\frac{\pi}{4}\right)}$$

$$= \frac{2\times 2}{-1} = -4$$

Given that equation whose roots are A and B is

Sum of the roots, $A + B = -3$,

Product of roots, $AB = -4$

Equation obtained is

$x^2 + 3x - 4 = 0$ and given equation is $ax^2 + bx - 4 = 0$.

On comparing

$\therefore \mathrm{a} = 1, \mathrm{b} = 3$

3. Number of the solutions of the equation $32^{\tan^2 x} + 32^{\sec^2 x} = 81,\ 0 \le x \le \frac{\pi}{4}$ is

Ans. 1

Sol. Given $32^{\tan^2 x} + 32^{\sec^2 x} = 81$

$32^{\tan^2 x} + 32^{1+\tan^2\ x} = 81$

Taking $32^{\tan^2 x}$ common

$33 \times 32^{\tan^2 x} = 81$

$32^{\tan^2 x} = \dfrac{81}{33}$

Taking log both sides with base 32

$\tan^2\ x = \log_{32}\left(\frac{27}{11}\right)$ as $0 \le x \le \frac{\pi}{4}$

$tan\ x = \sqrt{\log_{32}\left(\frac{27}{11}\right)} \in (0,1)$

one solution in $x \in \left[0, \frac{\pi}{4}\right]$.

4. Given the matrix $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, where $a, b, d \in \{-1,0,1\}$. If $(I - A)^3 = I - A^3$. Then the number of possible such matrix A are.

Ans. 8

Sol. Given $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, where $a, b, d \in \{-1,0,1\}$

$(I - A)^3 = I - A^3$

Applying formula of $(a - b)^3$ in LHS

$I - A^3 - 3A + 3A^2 = I - A^3$

$\Rightarrow 3\mathrm{A}^2 - 3\mathrm{A} = 0$

$\Rightarrow 3\mathrm{A}(\mathrm{A} - \mathrm{I}) = 0$

$\Rightarrow \mathrm{A}^2 = \mathrm{A}$

$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$

$\begin{bmatrix} a^2 & ab+bd \\ 0 & d^2 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$

On comparing both the matrices

$\Rightarrow a^2 = a \Rightarrow a = 0,1$

$d^2 = d \Rightarrow d = 0,1$

$b(a+d) = b$

$\Rightarrow b = 0 \; or \; a + d = 1$

Case I: $\mathrm{b} = 0 \Rightarrow (\mathrm{a}, \mathrm{d}) \equiv (0,1)(0,0),(1,1),(1,0) \rightarrow 4$ ways

Case II: $\mathrm{a} + \mathrm{d} = 1 \Rightarrow (1,0)(0,1)$ and $\mathrm{b} = \pm 1 \rightarrow 4$ ways

Total $= 8$ ways

5. Given $\frac{a_1+a_2+\cdots\ldots\ldots+a_{10}}{a_1+a_2+\cdots\ldots+a_p} = \frac{100}{P^2}$, then find the value of $\frac{a_{11}}{a_{10}} =$

(A) $\frac{19}{21}$

(B) $\frac{20}{21}$

(C) $\frac{20}{19}$

(D) $\frac{21}{19}$

Ans. (D)

Sol. Given $\frac{a_1+a_2+\cdots\ldots\ldots+a_{10}}{a_1+a_2+\cdots\ldots+a_p} = \frac{100}{P^2}$

As ratio of sum of 10 terms and sum of p terms is given

$\frac{\mathrm{S}_{10}}{\mathrm{S}_{\mathrm{P}}} = \frac{100}{\mathrm{P}^2}$

$\Rightarrow \mathrm{S}_{\mathrm{P}} = \frac{\mathrm{S}_{10} \cdot \mathrm{P}^2}{100} \ldots . (i)$

$\frac{a_{11}}{a_{10}} = \frac{S_{11}-S_{10}}{S_{10}-S_9}$

Using formula from (i)

$$= \frac{S_{10} \cdot \frac{121}{100} - S_{10}}{S_{10} - \frac{S_{10} \cdot 81}{100}}$$

$= \frac{21}{19}$

6. Given that the line $\frac{x-2}{\alpha} = \frac{y-3}{-5} = \frac{z-0}{2}$ is completely lies in the plane $x + 3y - 2z + \beta = 0$, then find the value of $(\alpha + \beta)$

(A) 2

(B) 4

(C) 8

(D) 16

Ans. (C)

Sol. Given that $\frac{x-2}{\alpha} = \frac{y-3}{-5} = \frac{z-0}{2}$ lies completely in plane.

Line passes through $(2,3,0)$ and having direction ratios as $(\alpha, -5, 2)$

The point $(2,3,0)$ lies in the plane $x + 3y - 2z + \beta = 0$.

$\Rightarrow 2 + 9 + \beta = 0$

$\Rightarrow \beta = -11$

$\because$ line lies in the plane so it is perpendicular to the normal of the plane.

Direction ratios of normal of the plane are $(1, 3, -2)$ and of line are $(\alpha, -5, 2)$.

Applying condition of perpendicularity i.e. sum of product of corresponding direction ratios is 0.

$\Rightarrow \alpha(1) - 5(3) + 2(-2) = 0$

$\Rightarrow \alpha = 19$

So, $\alpha + \beta = 8$

Question 7: If $S = \frac{7}{5} + \frac{9}{5^2} + \frac{13}{5^3} + \frac{19}{5^4} \ldots \ldots \infty,$ then the value of S is equal to

(A) $\frac{11}{32}$

(B) $\frac{61}{32}$

(C) $\frac{61}{33}$

(D) $\frac{61}{40}$

Ans (B)

Sol. Given

$S = \frac{7}{5} + \frac{9}{5^2} + \frac{13}{5^3} + \frac{19}{5^4} + \cdots \ldots \infty$ ....(1)

Its an A.G.P. with common ratio of G.P. is $\frac{1}{5}$

$\frac{S}{5} = \frac{7}{5^2} + \frac{9}{5^3} + \frac{13}{5^4} + \frac{19}{5^5} + \cdots . . \infty$ ....(2)

equation $(1) - (2)$

$\frac{4S}{5} = \frac{7}{5} + \frac{2}{5^2} + \frac{4}{5^3} + \frac{6}{5^4} + \frac{8}{5^5} + \cdots . \infty$

$\left(\frac{4S}{5} - \frac{7}{5}\right) = k = \frac{2}{5^2} + \frac{4}{5^3} + \frac{6}{5^4} + \frac{8}{5^5} + \cdots . \infty . . . . (3)$

Its again an A.G.P. with common ratio of G.P. is $\frac{1}{5}$

$\frac{k}{5} = \frac{2}{5^3} + \frac{4}{5^4} + \frac{6}{5^5} + \frac{8}{5^6} \ldots . \infty . . . . (4)$

equation $(4) - (3)$

$\frac{4\mathrm{k}}{5} = \frac{2}{5^2} + \frac{2}{5^3} + \frac{2}{5^4} + \frac{2}{5^5} \ldots . \infty$

Its an infinite G.P. with common ratio of G.P. is $\frac{1}{5}$.

$\frac{4k}{5} = \frac{2}{25}\left\{\frac{1}{1-\frac{1}{5}}\right\} = \frac{1}{10}$

$\Rightarrow k = \frac{1}{8}$

Now, $\frac{4S}{5} - \frac{7}{5} = \frac{1}{8}$

$\Rightarrow \frac{4S}{5} = \frac{7}{5} + \frac{1}{8}$

$\Rightarrow \mathrm{S} = \frac{61}{32}$

8. Given that a tangent to the parabola $y^2 = 8x$ at $(2, -4)$ also touches the circle $x^2 + y^2 = a$, then the value of a is

Ans. 2

Sol. Equation of tangent to parabola $y^2 = 8x$ at $(2, -4)$ is

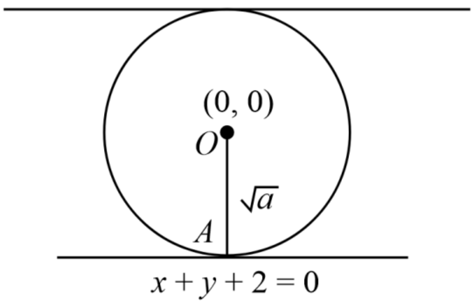
$T = 0$

$-4y = 4(x + 2)$

$\Rightarrow$ i.e. $x + y + 2 = 0$

This line is also tangent to the circle $x^2 + y^2 = a$

Centre $\mathrm{O} = (0,0)$ and radius $\mathrm{r} = \sqrt{a}$.



As we know that for a line to be tangent to a circle, perpendicular from centre of the circle is equal to radius of the circle.

$\mathrm{OA} = \sqrt{\mathrm{a}}$

$\Rightarrow \left|\frac{0+0+2}{\sqrt{2}}\right| = \sqrt{a}$

$\Rightarrow \sqrt{2} = \sqrt{a}$

$\Rightarrow a = 2$

9. If the mean & variance of the 7 observations is 8 & 12 respectively. Given that two of the observations are 8 and 6. Find the variance of the 5 remaining observations.

(A) $\frac{792}{25}$

(B) $\frac{396}{50}$

(C) $\frac{396}{25}$

(D) $\frac{132}{15}$

Ans. (C)

Sol. Given that mean & variance of the 7 observations is 8 & 12 respectively also two of the observations are 8 and 6.

Let a, b, c, d, e be 5 unknown observations.

$n = 7$, "Mean"$(\mu) = 8$, Variance $(\sigma^2) = 12$

$\therefore$ sum of observations $= n\mu = 7 \times 8 = 56$

$\Rightarrow$ Mean of 5 remaining observations $= \frac{56-8-6}{5}$

$= \frac{42}{5}$

Variance $(\sigma^2) = \frac{\Sigma x_i^2}{n} - \mu^2$

$12 = \frac{\Sigma x_i^2}{7} - 64$

$\Rightarrow \Sigma x_i^2 = 532$

$\Rightarrow a^2 + b^2 + c^2 + d^2 + e^2 + 64 + 36 = 532$

$\Rightarrow a^2 + b^2 + c^2 + d^2 + e^2 = 432$

$\therefore$ Variance of remaining 5 observations $= \frac{\Sigma x_i^2}{n} - \mu^2$

$$= \frac{432}{5} - \left(\frac{42}{5}\right)^2 = \frac{396}{25}$$

10. Given that the function $f(x) = \sin^{-1}\left(\frac{3x^2+x-1}{(x-1)^2}\right) + \cos^{-1}\left(\frac{x-1}{x+1}\right)$ then what is the domain of $f(x)$:

(A) $\left(0, \frac{1}{2}\right)$

(B) $\left[\frac{1}{4}, \frac{1}{2}\right] \cup \{0\}$

(C) $\left[\frac{1}{4}, \frac{1}{2}\right]$

(D) $\left(\frac{1}{4}, \frac{1}{2}\right)$

Ans. (C)

Sol. Given $f(x) = \sin^{-1}\left(\frac{3x^2+x-1}{(x-1)^2}\right) + \cos^{-1}\left(\frac{x-1}{x+1}\right)$

We know that $\sin^{-1}x$ & $\cos^{-1}x$ are defined when $-1 \le x \le 1$.

Now for $\sin^{-1}\left(\frac{3x^2+x-1}{(x-1)^2}\right)$ to be defined-

$$-1 \le \frac{3x^2 + x - 1}{(x-1)^2} \le 1$$

Denominator is positive so cross multiply can be done.

$-(x-1)^2 \le 3x^2 + x - 1 \le (x-1)^2$

$-(x-1)^2 \le 3x^2 + x - 1$ and $3x^2 + x - 1 \le (x-1)^2$

$4x^2 - x \ge 0$ and $2x^2 + 3x - 2 \le 0$

$x(4x-1) \ge 0$ and $(x+2)(2x-1) \le 0$

$x \in (-\infty, 0) \cup \left[\frac{1}{4}, \infty\right) \ldots . (i)$ *and* $x \in \left[-2, \frac{1}{2}\right] \ldots . . (ii)$

From $(i) \cap (ii)$

$\therefore x \in [-2,0] \cup \left[\frac{1}{4}, \frac{1}{2}\right] \ldots . (iii)$

And for $\cos^{-1}\left(\frac{x-1}{x+1}\right)$ to be defined-

$-1 \leq \frac{x-1}{x+1} \leq 1$

$-1 \leq \frac{x-1}{x+1} \leq 0$ and $\frac{x-1}{x+1} \leq 1$

$\frac{-2x}{x+1} \leq 0$ and $\frac{x-1}{x+1} - 1 \leq 0$

$\frac{x}{x+1} \geq 0$ and $\frac{-2}{x+1} \leq 0$

$x \in (-\infty, -1) \cup [0, \infty) \ldots (iv)$ and $x \in (-1, \infty) \ldots (v)$

From $(iv) \cap (v)$

$\therefore x \in [0, \infty) \ldots . (vi)$

From (iii) and (vi),

$(iii) \cap (vi)$

Domain is $x \in \{0\} \cup \left[\frac{1}{4}, \frac{1}{2}\right]$

11. Find the negation of the statement $(p \vee r) \rightarrow (q \vee r)$

(A) $p \wedge q \wedge r$

(B) $p \wedge \sim q \wedge \sim r$

(C) $p \wedge \sim q \wedge r$

(D) $\sim p \wedge q \wedge r$

Ans. (B)

Sol. Given that $(p \vee r) \rightarrow (q \vee r)$

$\equiv \sim ((p \vee r) \rightarrow (q \vee r))$

$\equiv (p \vee r) \wedge (\sim (q \vee r))$ (Applying De-Morgan's Law)

$\equiv (p \vee r) \wedge (\sim q \wedge \sim r)$ (Applying Associative Law)

$\equiv ((p \vee r) \wedge \sim r) \wedge (\sim q)$ (Applying Distributive Law)

$\equiv ((p \vee \sim r) \vee (r \wedge \sim r)) \wedge (\sim q)$ (Applying $p \wedge \sim p = f$)

$\equiv ((p \wedge \sim r) \vee f) \wedge (\sim q)$ (Applying $p \vee f = p$)

$\equiv (p \wedge \sim r) \wedge (\sim q)$

$\equiv p \wedge \sim q \wedge \sim r$

12. Given that $f: N \rightarrow N$ for which $f(m+n) = f(m) + f(n) \; \forall \; m, n \in N$. If $f(6) = 18$ then find the value of $f(2).f(3)$.

Ans. 2

Sol. Given $f(m+n) = f(m) + f(n) \; \forall \; m, n \in N$

Put $m = n = 3$

$f(3+3) = f(3) + f(3)$

$\Rightarrow f(6) = f(3) + f(3)$

$\Rightarrow 2f(3) = 18$

$\Rightarrow f(3) = 9$

Now, $f(3) = f(2+1) = f(2) + f(1)$

$\Rightarrow f(2) + f(1) = f(1+1) + f(1)$

$f(3) = f(1) + f(1) + f(1)$

$9 = 3f(1)$

$\Rightarrow f(1) = 3$

$f(2) = f(1+1) = f(1) + f(1) = 6$

$f(2).f(3) = 6 \times 9 = 54$

13. If $\vec{a} \times [(\vec{r} - \vec{b}) \times \vec{a}] + \vec{b} \times [(\vec{r} - \vec{c}) \times \vec{b}] + \vec{c} \times [(\vec{r} - \vec{a}) \times \vec{c}] = \vec{0}$ and $\vec{a}, \vec{b}$ and $\vec{c}$ are unit vector mutually perpendicular to each other then the value of $\vec{r}$ is:

(A) $\frac{\vec{a}-\vec{b}+\vec{c}}{2}$

(B) $\frac{\vec{a}-\vec{b}-\vec{c}}{2}$

(C) $\frac{\vec{a}+\vec{b}-\vec{c}}{2}$

(D) $\frac{\vec{a}+\vec{b}+\vec{c}}{2}$

Ans. (D)

Sol. Given that

$\vec{a} \times [(\vec{r} - \vec{b}) \times \vec{a}] + \vec{b} \times [(\vec{r} - \vec{c}) \times \vec{b}] + \vec{c} \times [(\vec{r} - \vec{a}) \times \vec{c}] = \vec{0}$

Applying formula of vector triple product $\vec{a} \times [\vec{b} \times \vec{c}] = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

$$(\vec{a}\cdot\vec{a})(\vec{r}-\vec{b})-(\vec{a}\cdot(\vec{r}-\vec{b}))\vec{a}+(\vec{b}\cdot\vec{b})(\vec{r}-\vec{c})-(\vec{b}\cdot(\vec{r}-\vec{c}))\vec{b}+(\vec{c}\cdot\vec{c})(\vec{r}-\vec{a})-(\vec{c}\cdot(\vec{r}-\vec{a}))\vec{c}=\vec{0}$$

As $\vec{a}, \vec{b}$ and $\vec{c}$ are unit vector mutually perpendicular to each other so their dot products

$\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0$ Also $|\vec{a}|^2=|\vec{b}|^2=|\vec{c}|^2=1$.

$\Rightarrow |\vec{a}|^2(\vec{r}-\vec{b})-(\vec{r}\cdot\vec{a})\vec{a}+|\vec{b}|^2(\vec{r}-\vec{c})-(\vec{r}\cdot\vec{b})\vec{b}+|\vec{c}|^2(\vec{r}-\vec{a})-(\vec{r}\cdot\vec{c})\vec{c}=\vec{0}$

$\Rightarrow \vec{r}-\vec{b}-(\vec{r}\cdot\vec{a})\vec{a}+\vec{r}-\vec{c}-(\vec{r}\cdot b)\vec{b}+\vec{r}-\vec{a}-(\vec{r}\cdot\vec{c})\vec{c}=\vec{0}$

$\Rightarrow 3\vec{r}-(\vec{a}+\vec{b}+\vec{c})-((\vec{r}\cdot\vec{a})\vec{a}+(\vec{r}\cdot\vec{b})\vec{b}+(\vec{r}\cdot\vec{c})\vec{c})=\vec{0}$

[Let $\vec{r}=x\vec{a}+y\vec{b}+z\vec{c}$]

$\Rightarrow (\vec{r}\cdot\vec{a})\vec{a}+(\vec{r}\cdot\vec{b})\vec{b}+(\vec{r}\cdot\vec{c})\vec{c}=\vec{r}$

$\therefore 3\vec{r}-(\vec{a}+\vec{b}+\vec{c})-\vec{r}=\vec{0}$

$\Rightarrow \vec{r}=\frac{\vec{a}+\vec{b}+\vec{c}}{2}$

14. If line pass through the point $(-3,-5)$ and intersect the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$ at point A and B. Then find the locus of mid-point of line joining A and B.

(A) $9x^2+4y^2+27x+20y=0$

(B) $9x^2+4y^2+20x+27y=0$

(C) $4x^2+9y^2+27x+20y=0$

(D) $4x^2+9y^2+20x+27y=0$

Ans. (A)

Sol. Given the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$and line passes through $(-3,-5)$.

Let mid-point of line AB is $P(h,k)$

So, Equation of chord AB is:

$\mathrm{T}=\mathrm{S}_1$

$\frac{hx}{4}+\frac{ky}{9}=\frac{h^2}{4}+\frac{k^2}{9}$

It passes through point $(-3,-5)$

$\frac{-3h}{4}-\frac{5k}{9}=\frac{h^2}{4}+\frac{k^2}{9}$

$\Rightarrow 9h^2+4k^2+27h+20k=0$

So, locus of $P(h,k)$ is

Replace $h \to x, k \to y$

$\Rightarrow 9x^2 + 4y^2 + 27x + 20y = 0$

15. Value of $\pi^2 \int_0^2 \sin\left(\frac{\pi x}{2}\right)(x - [x])^{[x]} dx$ is

(A) $4\pi + 2$

(B) $4\pi - 2$

(C) $4\pi + 4$

(D) $4\pi - 4$

Ans. D

Sol. Given $\pi^2 \int_0^2 \sin\left(\frac{\pi x}{2}\right)(x - [x])^{[x]} dx$

$$I = \pi^2 \int_0^2 \sin\left(\frac{\pi x}{2}\right)(x - [x])^{[x]} dx$$

Splitting the limit of integral where $[x]$ becomes integer and using the value of $[x]$.

$I = \pi^2 \int_0^1 \sin\left(\frac{\pi x}{2}\right) x^0 dx + \pi^2 \int_1^2 \sin\left(\frac{\pi x}{2}\right)(x-1)^1 dx$

$I = \pi^2 \int_0^1 \sin\left(\frac{\pi x}{2}\right) dx + \pi^2 \int_1^2 \sin\left(\frac{\pi x}{2}\right)(x-1) dx$

Applying standard integral formula of $\int \sin x \, dx$ and integration by parts respectively in both integrals

$I = \pi^2 \left[-\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right)\right]_0^1 + \pi^2 \left[\left((x-1)\frac{2}{\pi}\left(-\cos\left(\frac{\pi x}{2}\right)\right)\right)_1^2 - \int_1^2 \frac{2}{\pi}\left(-\cos\frac{\pi x}{2}\right) \mathrm{d}x\right]$

$I = \pi^2\left(\frac{2}{\pi}\right) + \frac{2\pi^2}{\pi}[1-0] + \frac{2\pi^2}{\pi} \times \frac{2}{\pi}\left(\sin\frac{\pi x}{2}\right)_1^2$

$I = 2\pi + 2\pi + 4(0-1)$

$I = 4\pi - 4$

16. If angle between the curves $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ & $x^2 + y^2 = ab, (a > b)$ is θ, then find the value of $\tan\ \theta$

(A) $\left|\frac{(a-b)}{ab}\right|$

(B) $\left|\frac{(a+b)}{ab}\right|$

(C) $\left|\frac{(a+b)}{\sqrt{ab}}\right|$

(D) $\left|\frac{(a-b)}{\sqrt{ab}}\right|$

Ans. (A)

Sol. Given

The angle between the curves $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ldots(i)$ & $x^2+y^2=ab\ldots.(ii)$ is θ

Solving equation (i) & (ii)

From (i) $b^2x^2+a^2y^2=a^2b^2\ldots(iii)$

Substitute from $(ii)\left(b^2x^2+a^2(ab-x^2)\right)=a^2b^2\ldots(iv)$

$x^2=\frac{ba^2(b-a)}{b^2-a^2}=\frac{a^2b}{a-b}$,

$$y^2=\frac{ab^2}{a+b}$$

Point of intersection is $\left(\pm\sqrt{\frac{a^2b}{a+b}},\pm\sqrt{\frac{ab^2}{a+b}}\right)$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

On differentiating wrt x

$$\Rightarrow \frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$\Rightarrow m_1=-\frac{b^2x}{a^2y}$

$x^2+y^2=ab$

On differentiating wrt x

$$\Rightarrow 2x+2y\cdot\frac{dy}{dx}=0$$

$\Rightarrow m_2=-\frac{x}{y}$

Angle between two curves is same as the angle between tangents at their point of intersection.

$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\left|\frac{\frac{-b^2x}{a^2y}+\frac{x}{y}}{1+\frac{b^2x^2}{a^2y^2}}\right|$

$=\left|\frac{xy(a^2-b^2)}{b^2x^2+a^2y^2}\right|=\left|\frac{xy(a^2-b^2)}{a^2b^2}\right|$

$=\left|\sqrt{\frac{a^3b^3}{(a+b)^2}}\cdot\frac{(a^2-b^2)}{a^2b^2}\right|=\left|\frac{(a-b)}{\sqrt{ab}}\right|$

17. Find the distance of point $P(-2,1,2)$ from the line of intersection of $x+3y-2z+1=0$ and $x-2y+z=0$.

(A) $\sqrt{\frac{34}{35}}$

(B) $\sqrt{\frac{35}{34}}$

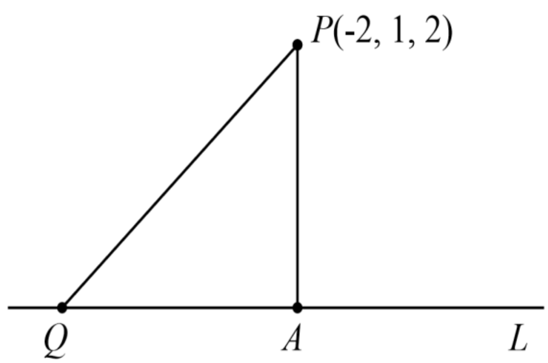
(C) $2 \times \sqrt{\frac{35}{34}}$

(D) $2 \times \sqrt{\frac{34}{35}}$

Ans. (D)

Sol. Given point is $P(-2,1,2)$

Line of intersection of $x + 3y - 2z + 1 = 0$ and $x - 2y + z = 0$



$\left(\frac{-2}{5}, \frac{-1}{5}, 0\right)$

Solving $x + 3y - 2z + 1 = 0$ and $x - 2y + z = 0$ using cross multiplication method

$$\frac{x}{3-4} = \frac{y}{-(1+2)} = \frac{z}{-2-3}$$

So, direction ratios are $(-1, -3, -5)$

Taking a point Q on plane by considering $z = 0$

Solving both equations $x + 3y + 1 = 0$ & $x - 2y = 0$

Point Q $\left(\frac{-2}{5}, \frac{-1}{5}, 0\right)$

Equation of the intersection line-

$$\frac{x+\frac{2}{5}}{-1} = \frac{x+\frac{1}{5}}{-3} = \frac{z-0}{-5}$$

As per diagram

$\mathrm{AP}^2 = \mathrm{PQ}^2 - \mathrm{QA}^2$

QA = Projection of PQ on line L

$$QA=\left|\frac{-1\left(-2+\frac{2}{5}\right)-3\left(1+\frac{1}{5}\right)-5(2-0)}{\sqrt{1+9+25}}\right|=\left|\frac{\frac{8}{5}-\frac{18}{5}-10}{\sqrt{35}}\right|$$

$$\mathrm{QA}=\frac{12}{\sqrt{35}}$$

$$\mathrm{PQ}=\sqrt{\left(\frac{-2}{5}+2\right)^2+\left(\frac{-1}{5}-1\right)^2+(0-2)^2}$$

$$\mathrm{PQ}=\sqrt{\left(\frac{64}{25}\right)+\left(\frac{36}{25}\right)+4}$$

$$\mathrm{AP}^2=\left(\frac{64}{25}\right)+\left(\frac{36}{25}\right)+4-\frac{144}{35}$$

$$\mathrm{AP}^2=\frac{136}{35}$$

$$\mathrm{AP}=2\times\sqrt{\frac{34}{35}}$$