JEE Main 2021 September 1 Shift 2 Mathematics

1. There are 15 points $P_1, P_2, P_3 \ldots \ldots$ on a circle. The number of possible triangles using the vertices P_i, P_j, P_k such that $i + j + k \neq 15$ is

Ans 443

Sol.

when $\mathrm{i} + \mathrm{j} + \mathrm{k} = 15$

case-I: $\mathrm{i} = 1, \mathrm{j} + \mathrm{k} = 14 \Rightarrow (2, 12)\ (3, 11)\ (4, 10)\ (5, 9)\ (6, 8) = 5$

case-II: $\mathrm{i} = 2, \mathrm{j} + \mathrm{k} = 13 \Rightarrow (3, 10)\ (4, 9)\ (5, 8)\ (6, 7) = 4$

case-III: $\mathrm{i} = 3, \mathrm{j} + \mathrm{k} = 12 \Rightarrow (4, 8)\ (5, 7) = 2$

case-IV: $\mathrm{i} = 4, \mathrm{j} + \mathrm{k} = 11 \Rightarrow (5, 6) = 1$

$\Rightarrow$ 12 ways

As, the points are lying on a circle. So, no three of them will be collinear. Hence, the number of possible triangles using the vertices P_i, P_j, P_k such that $i + j + k \neq 15$ is ${}^{15}\mathrm{C}_3 - 12 = 455 - 12 = 443$

2. If $f(x) = 3 + \cos^{-1}\left(\cos\left(\frac{\pi}{2} + x\right)\cos\left(\frac{\pi}{2} - x\right) + \sin\left(\frac{\pi}{2} + x\right)\sin\left(\frac{\pi}{2} - x\right)\right)$, then the minimum value of $f(x)$ is equal to

Ans 3

Sol.

$f(x) = 3 + \cos^{-1}\left(\cos\left(\frac{\pi}{2} + x\right)\cos\left(\frac{\pi}{2} - x\right) + \sin\left(\frac{\pi}{2} + x\right)\sin\left(\frac{\pi}{2} - x\right)\right)$

$\Rightarrow f(x) = 3 + \cos^{-1}\{-\sin(x).\sin(x) + \cos(x).\cos(x)\}$

$\Rightarrow f(x) = 3 + \cos^{-1}(-\sin^2 x + \cos^2 x)$

$\Rightarrow f(x) = 3 + \cos^{-1}(\cos 2x)$

As we know that $\cos^{-1}(y) \in [0, \pi]$. Hence, the minimum value of $f(x)$ is 3

3. If the value of $\cos^{-1}(\cos(-6)) + \sin^{-1}(\sin 5) - \tan^{-1}(\tan 2)$ is $\pi - k$, then the value of k is equal to

Ans. 3

Sol.

$\cos^{-1}(\cos(-6)) + \sin^{-1}(\sin 5) - \tan^{-1}(\tan 2)$

$= \cos^{-1}(\cos(6)) + \sin^{-1}(\sin 5) - \tan^{-1}(\tan 2)$

$= (2\pi - 6) + (5 - 2\pi) - (2 - \pi)$

$= \pi - 3$

4. Area bounded by the curves $y = |\cos x - \sin x|$ and $y = \cos x + \sin x$ between $x = 0$ and $x = \frac{\pi}{2}$, is

(A) $2\sqrt{2} - 2$ sq. units

(B) $4 - 2\sqrt{2}$ sq. units

(C) 4 sq. units

(D) $4 - \sqrt{2}$ sq. units

Ans (B)

Sol.

$A = \int_0^{\pi/2} ((\cos x + \sin x) - |\cos x - \sin x|)dx$

$\Rightarrow A = \int_0^{\pi/4} ((\cos x + \sin x) - (\cos x - \sin x))dx + \int_{\pi/4}^{\pi/2} ((\cos x + \sin x) - (\sin x - \cos x))dx$

$\Rightarrow A = 2\int_0^{\pi/4} \sin x dx + 2\int_{\pi/4}^{\pi/2} \cos x dx$

$\Rightarrow A = 2\left(-\frac{1}{\sqrt{2}} + 1\right) + 2\left(1 - \frac{1}{\sqrt{2}}\right) = 4 - 2\sqrt{2}$ sq. units

5. $\sim (p \rightarrow q)$ is equivalent to

(A) $q \rightarrow (p \wedge q)$

(B) $p \rightarrow (p \vee q)$

(C) $\sim (p \rightarrow (p \rightarrow q))$

(D) Fallacy

Ans (C)

Sol. We know that $\sim (p \rightarrow q) \equiv p \wedge \sim q$ and $(p \rightarrow q) \equiv \sim p \vee q$ now using these

(A)

$q \rightarrow (p \wedge q)$

$\equiv \sim q \vee (p \wedge q)$

$\equiv (\sim q \vee p) \wedge (\sim q \vee q)$

$\equiv (\sim q \vee p) \wedge t$

$\equiv (\sim q \vee p)$

$\equiv q \rightarrow p$

$(B)\ p \rightarrow (p \vee q) \equiv \sim p \vee (p \vee q)$

$\equiv (\sim p \vee p) \vee q$

$\equiv t \vee q$

$\equiv t$

$(C)\ \sim (p \rightarrow (p \rightarrow q)) \equiv p \wedge \sim (p \rightarrow q)$

$\equiv p \wedge (p \wedge \sim q)$

$\equiv (p \wedge p) \wedge \sim q$

$\equiv p \wedge \sim q$

$\equiv \sim (p \rightarrow q)$

6. If $x^2dy + \left(y - \frac{1}{x}\right)dx = 0$ and $y(1) = 1$ then the value of $y(0.5)$ is equal to

(A) $e - 3$

(B) $3 + e$

(C) $3 - e$

(D) e^2

Ans (C)

Sol. $x^2dy + ydx = \frac{dx}{x}$

$\Rightarrow \frac{dy}{dx} + \frac{y}{x^2} = \frac{1}{x^3}$

I.F. $= e^{\int \frac{1}{x^2}dx} = e^{-\frac{1}{x}}$

$\Rightarrow y \cdot e^{-\frac{1}{x}} = \int e^{-\frac{1}{x}} \cdot \frac{1}{x^3}dx + C$

Let $\frac{-1}{x} = t \Rightarrow \frac{1}{x^2}dx = dt$

$\Rightarrow y \cdot e^{-\frac{1}{x}} = \int -te^t \cdot dt + C$

$\Rightarrow y \cdot e^{-\frac{1}{x}} = -[te^t - e^t] + C$

$\Rightarrow y \cdot e^{-\frac{1}{x}} = \frac{e^{-\frac{1}{x}}}{x} + e^{-\frac{1}{x}} + C$

Now, using $\mathrm{y}\ (1) = 1$, we get

$\Rightarrow (1) \cdot \mathrm{e}^{-1} = \frac{\mathrm{e}^{-1}}{1} + \mathrm{e}^{-1} + \mathrm{C}$

$\Rightarrow \mathrm{C} = -\mathrm{e}^{-1}$

Hence, the equation of curve is

$y \cdot e^{-\frac{1}{x}} = \frac{1}{x} e^{-\frac{1}{x}} + e^{-\frac{1}{x}} - e^{-1}$

$\Rightarrow y = \frac{1}{x} + 1 - \frac{e^{\frac{1}{x}}}{e}$

Now, $x = \frac{1}{2} \Rightarrow y\left(\frac{1}{2}\right) = 2 + 1 - \frac{e^2}{e} \Rightarrow y = 3 - e$

7. If the sum of the binomial coefficients of the expression $(x+y)^{\mathrm{m}}$ is 4096 then the value of greatest binomial coefficient is

(A) ${}^{12}\mathrm{C}_5$

(B) ${}^{12}\mathrm{C}_6$

(C) ${}^{13}\mathrm{C}_6$

(D) ${}^{14}\mathrm{C}_7$

Ans (B)

Sol. Sum of binomial coefficients of the expression $(x+y)^m = 2^m = 4096 = 2^{12} \Rightarrow m = 12$

now greatest binomial coefficient of the expression $(x+y)^{\mathrm{m}}$ will be the coefficient of its middle term, which is ${}^{\mathrm{m}}\mathrm{C}_6 = {}^{12}\mathrm{C}_6$

8. If $f(x)$ is a cubic polynomial such that $f(x) = \frac{-2}{x}$ for $x = 2,3,4$ and 5, then the value of $f(10)$ is equal to

Ans 2.6

Sol.

As, $f(x) = \frac{-2}{x}$ for $x = 2,3,4$ and 5

$\Rightarrow xf(x) + 2 = a(x-2)(x-3)(x-4)(x-5) \ldots (i)$

Put $x = 0$

$\Rightarrow 2 = a(-2)(-3)(-4)(-5)$

$\Rightarrow a = \frac{1}{60}$

Put a $= \frac{1}{60}$ in (i), we get

$xf(x) + 2 = \frac{1}{60}(x-2)(x-3)(x-4)(x-5)$

Now, putting $x = 10$, we get

$10f(10) + 2 = \frac{1}{60} \times 8 \times 7 \times 6 \times 5$

$\Rightarrow 10f(10) = 26$

$\Rightarrow f(10) = 2.6$

9. If $\lim_{x \to \frac{\pi}{4}} \frac{\frac{\pi}{4}\int_2^{\sec^2 x} f(x)dx}{x^2 - \frac{\pi^2}{16}} = kf(2)$, then the value of k is equal to

Ans 2

Sol. Using LHopital's rule, $\lim_{x \to \frac{\pi}{4}} \frac{\frac{\pi}{4} \cdot 2 \cdot \sec x \cdot \sec x \cdot \tan x \cdot f(\sec^2 x) - 0}{2x}$

$= \frac{\frac{\pi}{4} \cdot 2(\sqrt{2})^2 \cdot (1) \cdot f(2)}{2 \cdot \frac{\pi}{4}} = 2f(2)$

10. If the angle between the ellipse $\frac{x^2}{9} + \frac{y^2}{1} = 1$ and the circle $x^2 + y^2 = 3$ at their point of intersection in $1st$ quadrant is θ, then the value of $\sqrt{3}\tan\theta$ is equal to

Ans 2

Sol. The point of intersection of the curves $\frac{x^2}{9} + \frac{y^2}{1} = 1$ and $x^2 + y^2 = 3$ in the first quadrant is $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$ by solving these two equations.

Now slope of tangent to the ellipse $\frac{x^2}{9} + \frac{y^2}{1} = 1$ at $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right) = m_1 = -\frac{1}{3\sqrt{3}}$

and slope of tangent to the circle $x^2 + y^2 = 3$ at $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right) = m_2 = -\sqrt{3}$

So, if angle between both curves is θ then $\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right| = \left|\frac{-\frac{1}{3\sqrt{3}}+\sqrt{3}}{1+\left(-\frac{1}{3\sqrt{3}}\right)(-\sqrt{3})}\right| = \left(\frac{2}{\sqrt{3}}\right)$

11. If $a_1, a_2, a_3 \ldots \ldots a_{21}$ are in increasing A.P. such that $\sum_{n=1}^{20} \frac{1}{a_n \cdot a_{n+1}} = \frac{4}{9}$ and sum of these 21 terms is 189,

then the value of $a_6 \cdot a_{16}$ is

(A) 36

(B) 72

(C) 144

(D) 288

Ans (B)

Sol. $\sum_{n=1}^{20} \frac{1}{a_n a_{n+1}} = \sum \frac{1}{a_n(a_n+d)} = \frac{4}{9}$

$$= \frac{1}{d}\sum_{n=1}^{20}\left(\frac{1}{a_n} - \frac{1}{a_n+d}\right) \Rightarrow \frac{1}{d}\left[\left(\frac{1}{a_1}-\frac{1}{a_2}\right)+\left(\frac{1}{a_2}-\frac{1}{a_3}\right)+\cdots.\left(\frac{1}{a_{20}}-\frac{1}{a_{21}}\right)\right] = \frac{4}{9}$$

$$\Rightarrow \frac{1}{d}\left(\frac{1}{a_1} - \frac{1}{a_{21}}\right) = \frac{4}{9}$$

$$\Rightarrow \frac{1}{d}\left(\frac{a_{21}-a_1}{a_1 \cdot a_{21}}\right) = \frac{4}{9}$$

Using $a_{21} = a_1 + 20d$

$\Rightarrow a_1 a_{21} = 45$

$\Rightarrow a_1(a_1 + 20d) = 45 \ldots.(1)$

Now using sum of first 21 terms

$$\Rightarrow \frac{21}{2}(2a_1 + 20d) = 189$$

$\Rightarrow a_1 + 10d = 9 \ldots.(2)$

by using equation (1) and (2), we get

$a_1 = 3, d = \frac{3}{5}$

and $a_1 = 15, d = -\frac{3}{5}$ which is not possible as increasing A.P.

Hence, $a_6 \cdot a_{16} = (a_1 + 5d)(a_1 + 15d) = 72$

12. The number of words that can be formed using all the letters of word "FARMER" such that both R do not appear together, is

Ans 240

Sol. Number of required words = Total words − Words in which both R appears together

$=\frac{6!}{2!}-5!$

$=360-120$

$=240$

13. The area of the triangle formed by the lines $2x-y+1=0, 3x-y+5=0$ and $2x-5y+11=0$ is:

(A) $\frac{361}{52}$

(B) $\frac{362}{52}$

(C) $\frac{363}{52}$

(D) $\frac{364}{52}$

Ans (A)

Sol.

Intersection point of the lines

$2x-y+1=0$

and $3x-y+5=0$

is $(-4,-7)$

Intersection points of the lines

$2x-y+1=0$ and $2x-5y+11=0$ is $\left(\frac{3}{4},\frac{5}{2}\right)$

Similarly, intersection points of the lines

$3x-y+5=0$ and $2x-5y+11=0$ is

$\left(\frac{-14}{13},\frac{23}{13}\right)$

So, the area $=\left|\frac{1}{2}\begin{vmatrix}\frac{3}{4} & \frac{5}{2} & 1\\ -4 & -7 & 1\\ \frac{-14}{13} & \frac{23}{13} & 1\end{vmatrix}\right|=\frac{361}{52}$